Monads and Graphs*

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1 Introduction

The simplicial category Δ plays many roles all over mathematics. In one of these roles it is a monad freely generated by a single object (see [9], Section 3, [2], Section 4, and references therein). That Δ is isomorphic to this free monad may be understood as a coherence result connecting the syntax brought by the equational presentation of monads and the semantics given by the order preserving functions on finite ordinals, which can be graphically presented as pictures, or diagrams, or graphs, made of threads connecting the points at the top of the picture with some points at the bottom. However, this isomorphism is more than just coherence, for which it would be enough to have the faithfulness of a functor going from the free monad to the graphical category Δ.

An analogous result, which may be understood as a coherence result in the sense of category theory, is the isomorphism of the commutative Frobenius monad freely generated by a single object with the skeleton of the category *2Cob*, whose arrows are cobordisms in dimension 2 (see [1] and [13]). Still another example of such a result is the relation between commutative separable Frobenius monads and the category $\mathbf{Cospan}(\mathbf{Sets}_{fin})$, whose arrows are cospans in the base category *Set* (see [14] and [16]).

In logical terms, this is like proving completeness with respect to a manageable model, which helps us to solve the decision problem for commuting of diagrams of arrows. The categories we envisage that serve as manageable models are either categories whose arrows are of a geometric kind (tangles, oriented tangles, Temperley-Lieb diagrams, Kelly-Mac Lane graphs, Brauer diagrams),

*This is an extended abstract of the papers [10] and [11].

or they are obtained from such categories by abstraction. This is how we obtain the manageable categories *SplPre*, *Gen* and *Rel* whose objects are always the finite ordinals, and whose arrows are, respectively, preordering relations, equivalence relations and all binary relations with a specified way of composing (gluing) them.

The aim of this note is to complete the following table of corresponding pairs of notions, behind which we find coherence isomorphisms:

monad	Δ
commutative Frobenius monad	Skeleton of *2Cob*
commutative separable Frobenius monad	$\mathbf{Cospan}(\mathbf{Sets}_{fin})$
Frobenius monad	
	SplPre
	Gen
	Rel

In the next two sections we will define notions that occur in this table, and other notions that we need to fill the missing entries in the table.

2 Some Frobenius and bialgebraic monads

In this section we define various notions of monad and comonad, which occur on the left hand-side of the table above, or will serve to fill the missing entries on the left-hand side.

A *Frobenius monad* is given by a category $\mathcal{A}$ and an endofunctor M of $\mathcal{A}$ such that $\langle \mathcal{A}, M, \nabla, ! \rangle$ is a monad, $\langle \mathcal{A}, M, \Delta, ¡ \rangle$ is a comonad, and moreover the *Frobenius equations*, connecting the monad and comonad structures, are satisfied:

$$M\nabla \circ \Delta_M = \Delta \circ \nabla = \nabla_M \circ M\Delta.$$

A *commutative* Frobenius monad has moreover a natural symmetry isomorphism

$$\tau\colon MM \stackrel{.}{\to} MM,$$

inverse to itself, which satisfies besides the *Yang-Baxter equation*

$$M\tau \circ \tau_M \circ M\tau = \tau_M \circ M\tau \circ \tau_M,$$

the following *symmetrization equations*, connecting τ with the monad and comonad structures:

$$\begin{aligned} &\nabla \circ \tau = \nabla, && \tau \circ \Delta = \Delta, \\ &\tau \circ \nabla_M = M\nabla \circ \tau_M \circ M\tau, && \Delta_M \circ \tau = M\tau \circ \tau_M \circ M\Delta, \\ &\tau \circ\ !_M = M!, && ¡_M \circ \tau = M¡. \end{aligned}$$

The two symmetrization equations in the first line are the *commutativity equations.* A commutative Frobenius monad is *separable* when the following *separability equation* holds:

$$\nabla \circ \Delta = \mathbf{1}_M.$$

An *equivalential Frobenius monad* is a separable commutative Frobenius monad that satisfies moreover the following *unit-counit homomorphism equation*, appropriate for bialgebras:

$$¡ \circ ! = \mathbf{1}.$$

This equation is analogous to the separability equation.

A *preordering Frobenius monad* is an equivalential Frobenius monad that has an additional natural transformation

$$\downarrow : M \stackrel{\cdot}{\rightarrow} M,$$

which satisfies, first, the $\downarrow$*-idempotence equation*:

$$\downarrow \circ \downarrow = \downarrow,$$

and the following additional *symmetrization equation*:

$$\tau \circ \downarrow_M = M\!\downarrow \circ \tau.$$

With the definition $\uparrow =_{df} M¡ \circ M\nabla \circ M\!\downarrow_M \circ \Delta_M \circ !_M$, we have the *up-and-down equation*:

$$\nabla \circ M\!\downarrow \circ \uparrow_M \circ \Delta = \mathbf{1}_M.$$

This equation is analogous up to a point to the separability equation. With the definitions

$$\nabla^{\downarrow} =_{df} \nabla \circ M\!\downarrow \circ \downarrow_M, \qquad\qquad \Delta^{\downarrow} =_{df} M\!\downarrow \circ \downarrow_M \circ \Delta,$$

we have the three bialgebraic *multiplication-comultiplication homomorphism equations*, which for short we call the *mch equations*:

(2·0) $\quad ¡ \circ \nabla^{\downarrow} = ¡ \circ M¡,$ $\qquad$ (0·2) $\quad \Delta^{\downarrow} \circ ! = M! \circ !,$

(2·2) $\quad \Delta^{\downarrow} \circ \nabla^{\downarrow} = M\nabla^{\downarrow} \circ \nabla^{\downarrow}_{MM} \circ M\tau_M \circ MM\Delta^{\downarrow} \circ \Delta^{\downarrow}_M.$

Finally, we have one more equation involving bialgebraic multiplication and comultiplication, i.e. $\nabla^{\downarrow}$ and $\Delta^{\downarrow}$, which we call *bialgebraic separability*:

$$\nabla^{\downarrow} \circ \Delta^{\downarrow} = \downarrow$$

This equation is analogous to the separability equation given above for Frobenius multiplication and comultiplication, i.e. ∇ and Δ. This concludes the definition of a preordering Frobenius monad.

We call *commutative bialgebraic monad* a structure given by a category $\mathcal{A}$, an endofunctor $M^{\downarrow}$ of $\mathcal{A}$, and the natural transformations

$$\nabla^{\downarrow} \colon M^{\downarrow}M^{\downarrow} \stackrel{\cdot}{\to} M^{\downarrow}, \qquad \Delta^{\downarrow} \colon M^{\downarrow} \stackrel{\cdot}{\to} M^{\downarrow}M^{\downarrow},$$
$$! \colon I_{\mathcal{A}} \stackrel{\cdot}{\to} M^{\downarrow}, \qquad ¡ \colon M^{\downarrow} \stackrel{\cdot}{\to} I_{\mathcal{A}}$$

such that $\langle \mathcal{A}, M^{\downarrow}, \nabla^{\downarrow}, ! \rangle$ is a monad, $\langle \mathcal{A}, M^{\downarrow}, \Delta^{\downarrow}, ¡ \rangle$ is a comonad; moreover, we have a natural symmetry isomorphism

$$\tau \colon M^{\downarrow}M^{\downarrow} \stackrel{\cdot}{\to} M^{\downarrow}M^{\downarrow},$$

inverse to itself, which satisfies the Yang-Baxter equation and the symmetrization equations given above for ∇, Δ, ! and ¡ with the superscript $^{\downarrow}$ added to M, ∇ and Δ, and, finally, we have the four bialgebraic homomorphism equations given above with the superscript $^{\downarrow}$ added to M. This defines commutative bialgebraic monads. The category $\mathbf{L}(\mathbf{Z_2})$ of [15] (Section 3, see Figure 13) is isomorphic to the commutative bialgebraic monad that satisfies $\nabla^{\downarrow} \circ \Delta^{\downarrow} = ! \circ ¡$ freely generated by a single object.

A *relational bialgebraic monad* is a commutative bialgebraic monad that satisfies moreover the following version of the bialgebraic separability equation:

$$\nabla^{\downarrow} \circ \Delta^{\downarrow} = \mathbf{1}_{M^{\downarrow}}.$$

Let *Frob*, $\mathcal{PF}$, $\mathcal{EF}$and $\mathcal{RB}$ be the categories of respectively the Frobenius monad, the preordering Frobenius monad, the equivalential Frobenius monad, and the relational bialgebraic monad, all these monads being freely generated by a single object.

3 The categories *SplPre*, *Gen* and *Rel*

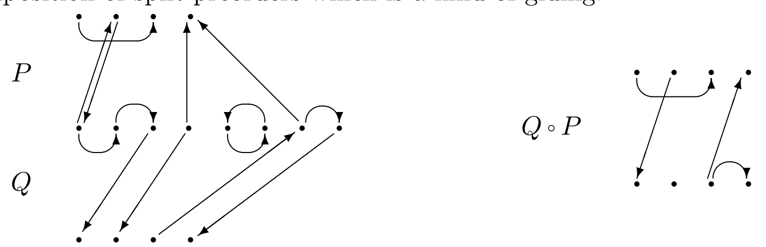
In this section we define first the categories that occur on the right-hand side of the last three lines of the table at the end of Section 1. The categories *Gen* and *Rel* are important for us because we have relied on these categories in our work on categorial coherence for various fragments of logic, and related structures (see [6], [7], [8], [9], [10], and references therein). The interest of the category *SplPre* in this perspective is that it is a common, natural, extension of both *Rel* and *Gen*.

It is easy to define precisely the category *Rel*, and we will do that first. Its objects are the finite ordinals, its arrows are the binary relations between finite ordinals, and composition of these arrows is the usual composition of relations:

$$R_2 \circ R_1 = \{(x, y) \mid \exists z((x, z) \in R_1 \;\; \& \;\; (z, y) \in R_2)\}.$$

It is very well known that this composition is associative, and that, with identity arrows being identity relations, *Rel* is a category.

The precise definition of the categories *SplPre* and *Gen* is a more involved matter. The objects of the category *SplPre* are again the finite ordinals and the arrows of this category, which we call *split preorders*, are preordering (i.e. reflexive and transitive) relations on the disjoint union of their sources and targets. We illustrate a split preorder $P : n \to m$ by a picture having n dots at the top and m dots at the bottom (taking them together as the disjoint union $n + m$) and with oriented threads connecting pairs belonging to P. We omit, however, to draw loops that correspond to the pairs (x, x) in P. See [5] and [11] for formal definitions of the composition in *SplPre*, and for proofs that this composition is associative. We give here just an illustrated example of composition of split preorders which is a kind of gluing:



There is an injection from the arrows of *Rel* to those of *SplPre*, which maps a binary relation $R \subseteq n \times m$ to the split preorder from n to m given by:

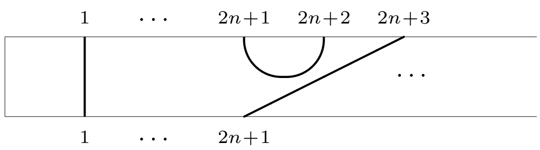
$$\{((i,1),(j,2)) \mid (i,j) \in R\} \;\cup\; \{((i,1),(i,1)) \mid i \in n\} \;\cup\; \{((j,2),(j,2)) \mid j \in m\}.$$

This injection preserves composition, but does not preserve identity arrows.

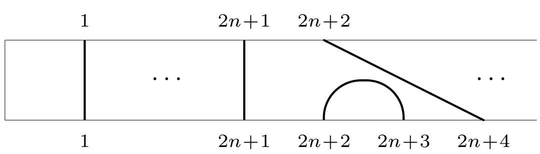
The category *Gen* is the subcategory of *SplPre* whose objects are the objects of *SplPre* and whose arrows are those split preorders that are symmetric; i.e., they are equivalence relations. We call such split preorders *split equivalences*. (This category was investigated in [4], where it was named *Gen* because of its connection with *generality* of proofs.)

The arrows of the category *Frz* are diagrams called *friezes*. Roughly speaking, a frieze is a tangle without crossings in whose regions we find circular forms that correspond bijectively to the ordinals contained in the infinite ordinal ε_0. According to Proposition 2 of [10] and results of [3], the categories *Frob* and *Frz* are isomorphic. By this isomorphism, the arrows on the left are mapped to the friezes on the right:

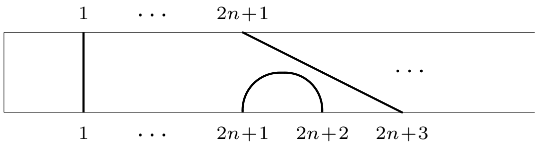
$i_n : n{+}1 \to n$



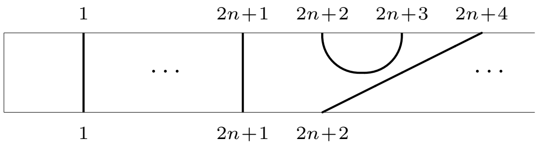
$\Delta_n : n{+}1 \to n{+}2$



$!_n : n \to n+1$



$\nabla_n : n+2 \to n+1$

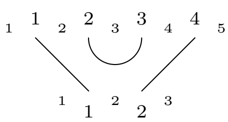


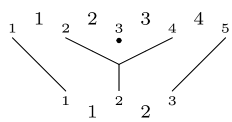
Note that our friezes are "thin" tangles, which may be conceived as the boundaries of the corresponding *thick* tangles of [12].

A *circular form* is a finite collection of nonintersecting circles in the plane factored through homeomorphisms of the plane mapping one collection into another (see the definition of $\mathcal{L}$-equivalence of friezes in [3], Section on *Friezes*). The circular forms obtained by composing friezes are coded by the ordinals contained in ε_0 in the following way. The circular form consisting of no circles is coded by 0. If the circular forms c_1, c_2 and c are coded by the ordinals α_1, α_2 and α respectively, then the circular form c_1c_2 (the disjoint union of c_1 and c_2) is coded by the natural sum $\alpha_1 \sharp \alpha_2$, and the circular form ⓒ (c inside a new circle) is coded by ω^α. So a single circle is coded by ω^0, which is equal to 1 (see [3], Section on *Finite multisets, circular forms and ordinals*).

Let $\mathcal{F}$ be the commutative monoid with one unary operation freely generated by the empty set of generators. The elements of $\mathcal{F}$ may be identified with the hierarchy of finite multisets obtained by starting from the empty multiset as the only urelement, or by finite nonplanar trees with arbitrary finite branching, or by circular forms. A monoid isomorphic to $\mathcal{F}$ is the commutative monoid $\langle \varepsilon_0, \sharp, 0, \omega^- \rangle$ where $\sharp$ is binary natural sum, and we have the additional unary operation ω^- (for more details on these matters, see [3]).

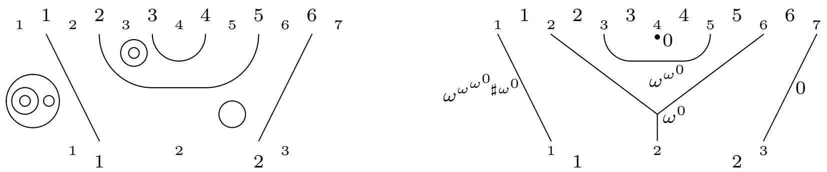
Instead of the category *Frz*, one can use an alternative isomorphic category, which we call *Frobse*. In the arrows of this category, the regions of friezes stand for some particular kind of split equivalences. For example, instead of the frieze on the left-hand side, which is an arrow of *Frz* of the type $2+l \to 1+l$, we have the nonintersecting split equivalence on the right-hand side, which is an arrow of *Frobse* of the same type:





The thick white regions on the left-hand side become thin black equivalence classes on the right-hand side, and the thin black threads on the left-hand side become white regions on the right-hand side. We will not obtain in this way on the right-hand side every split equivalence, but only those that satisfy some additional conditions, and are called *maximal* (see [10], Section 6). In the regions of friezes one finds finitely many circular forms that correspond to ordinals in ε_0, and we will assign these ordinals to the equivalence classes of maximal split equivalences.

Maximal split equivalences together with a function assigning ordinals in ε_0 to the equivalence classes, so that all but finitely many have zero as value, will be called *Frobenius* split equivalences. Frobenius split equivalences with types associated to them are the arrows of *Frobse*. For example, to the frieze on the left-hand side we assign the Frobenius split equivalence on the right-hand side:



The categories *Frob*, *Frz* and *Frobse* are all isomorphic, so that the missing entry on the right-hand side in the fourth line of the table at the end of Section 1 can be filled either by *Frz* or by *Frobse*. It remains to fill the missing entries on the left-hand side of the last three lines in this table, and, as it is shown in [11], they are filled respectively by "preordering Frobenius monad", "equivalential Frobenius monad", and "relational bialgebraic monad", which correspond to the categories $\mathcal{PF}$, $\mathcal{EF}$ and $\mathcal{RB}$.

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