Postulates for logic-based argumentation systems

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Abstract. This paper studies abstract logic-based argumentation systems. It proposes three key rationality postulates that such systems should satisfy: consistency, closure under sub-arguments and closure under the consequence operator of the underlying logic. It then investigates the links between these postulates, and explores the conditions under which they are guaranteed or violated.

1 Introduction

An *argumentation system* for defeasible reasoning consists of a set of *arguments*, *attacks* among them, and a *semantics* for evaluating the arguments. Indeed, acceptable sets of arguments, called *extensions*, are computed under a semantics. Arguments are built from a *knowledge base* using an *underlying logic*. A logic contains two parts: a *language* in which the formulas of the knowledge base are encoded, and a *consequence operator* which is used for defining arguments and attacks. In the ASPIC argumentation system [3], for instance, the language of its logic is made of two types of rules: strict rules which encode certain knowledge and defeasible rules which encode uncertain ones. The consequence operator shows how these rules can be chained. We will refer to such a logic as *rule-based logic* and to systems grounded on it as *rule-based systems*.

The first work on rationality postulates in argumentation was done by Caminada and Amgoud [10]. The authors focused *only* on rule-based systems, and proposed the following postulates that such systems should satisfy:

Closure: The idea is that if a system concludes x and there is a strict rule $x \to y$, then the system should also conclude y.

Direct consistency: the set of conclusions of arguments of each extension should be consistent.

Indirect consistency: the closure of the set of conclusions of arguments of each extension should be consistent.

As obvious as they may appear, these postulates are violated by most rule-based systems (like [17]). Besides, they are tailored for rule-based logics. Their counterparts for any other logic do not exist. Later, Amgoud and Besnard [2] made a first attempt on generalizing the two postulates on consistency to any logic. For that purpose, they considered abstract logics as defined by Tarski [19]. They defined a new postulate for direct consistency which is stronger than the original one. It imposes that the set of formulas that are used in the supports of arguments of each extension should be consistent. The authors justified this choice by the fact that an extension represents a coherent position/point of view, thus it should only involve a consistent set of formulas. They have then shown that indirect consistency follows naturally from the new postulate, thus indirect consistency does not deserve to be a postulate per se.

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As in [2], in this paper we consider argumentation systems that are grounded on Tarski's logics. We generalize the postulates that are proposed in [10] to any logic, and define a new one. The new postulate says that if an extension contains an argument, then all its sub-arguments should belong to the extension as well. We show that the strong version of direct consistency that is proposed in [2] follows naturally from the new postulate on sub-arguments and the extended version of the initial definition of direct consistency. Thus, strong consistency does not deserve to be a separate postulate. In sum, there are three basic postulates: 1) Closure under the consequence operator of the logic; 2) Closure under sub-arguments; 3) Direct consistency, i.e., the version defined in [10]. Indirect consistency and strong consistency follow from these postulates. We show that these postulates are *independent* and *compatible*. A second contribution of this paper consists of studying under which conditions the postulates are satisfied or violated. The satisfaction/violation of a postulate depends mainly on the attack relation. We characterize some attack relations that lead to the satisfaction of the three postulates, and some other relations that lead to the violation of consistency.

The paper is organized as follows: Section 2 defines the logic-based argumentation systems we are interested in. Section 3 introduces three basic postulates, and studies the links between them. Section 4 investigates the conditions under which the postulate on consistency is violated. The conditions under which the three postulates are satisfied are studied in Section 5. Section 6 discusses the importance of our postulates in case of weighted argumentation systems.

2 Logic-based Argumentation Systems

It is well known that a structured argumentation system is built on an underlying monotonic logic. In this paper, we do not focus on a particular logic (like rule-based logic, propositional logic, ...), but we consider an *abstract* monotonic logic. Such abstraction makes our study general and our results hold under any instantiation of the abstract logic. We consider Tarski's logics $(\mathcal{L}, \mathsf{CN})$ where $\mathcal{L}$ is a set of well-formed *formulas*. Note that there is no particular requirement on the kind of connectors that may be used. CN is a *consequence operator*. It is a function from $2^{\mathcal{L}}$ to $2^{\mathcal{L}}$ which returns the set of formulas that are logical consequences of another set of formulas according to the logic in question. It should satisfy the following basic properties:

1. $X \subseteq \mathsf{CN}(X)$ (Expansion)
2. $\mathsf{CN}(\mathsf{CN}(X)) = \mathsf{CN}(X)$ (Idempotence)
3. $\mathsf{CN}(X) = \bigcup_{Y \subseteq_f X} \mathsf{CN}(Y)$[2] (Finiteness)
4. $\mathsf{CN}(\{x\}) = \mathcal{L}$ for some $x \in \mathcal{L}$ (Absurdity)

[2] $Y \subseteq_f X$ means that Y is a finite subset of X.

5. $\mathsf{CN}(\emptyset) \neq \mathcal{L}$ (Coherence)

Any logic whose CN satisfies the above properties is *monotonic*. The associated notion of consistency is defined as follows:

Definition 1 (Consistency) *A set $X \subseteq \mathcal{L}$ is* consistent *wrt a logic $(\mathcal{L}, \mathsf{CN})$ iff $\mathsf{CN}(X) \neq \mathcal{L}$. It is* inconsistent *otherwise.*

Arguments are built from a *finite knowledge base* $\Sigma \subseteq \mathcal{L}$ as follows:

Definition 2 (Argument) *Let Σ be a knowledge base. An* argument *is a pair (X, x) s.t. $X \subseteq \Sigma$, X is consistent, and $x \in \mathsf{CN}(X)$*[3]. *An argument (X, x) is a* sub-argument *of another argument (X', x') iff $X \subseteq X'$.*

Notations: $\mathtt{Supp}$ and $\mathtt{Conc}$ denote respectively the *support* X and the *conclusion* x of an argument (X, x). For all $\mathcal{S} \subseteq \Sigma$, $\mathtt{Arg}(\mathcal{S})$ denotes the set of all arguments that can be built from $\mathcal{S}$ by means of Definition 2. $\mathtt{Sub}$ is a function that returns all the sub-arguments of a given argument. For all $\mathcal{E} \subseteq \mathtt{Arg}(\Sigma)$, $\mathtt{Concs}(\mathcal{E})$ = $\{\mathtt{Conc}(a) \mid a \in \mathcal{E}\}$ and $\mathtt{Base}(\mathcal{E})$ = $\bigcup_{a \in \mathcal{E}} \mathtt{Supp}(a)$. Let $\mathcal{C}_\Sigma$ denote the set of all *minimal conflicts*[4] of Σ.

An *argumentation system* is defined as follows:

Definition 3 (Argumentation system) *An* argumentation system *(AS) over a knowledge base Σ is a pair $(\mathtt{Arg}(\Sigma), \mathcal{R})$ where $\mathcal{R} \subseteq \mathtt{Arg}(\Sigma) \times \mathtt{Arg}(\Sigma)$ is an* attack *relation. For $a, b \in \mathtt{Arg}(\Sigma)$, $(a, b) \in \mathcal{R}$ (or $a\mathcal{R}b$) means that a* attacks *b.*

The attack relation is left *unspecified* in order to keep the system very general. It is also worth mentioning that the set $\mathtt{Arg}(\Sigma)$ may be infinite even when the base Σ is finite. This would mean that the argumentation system may be *infinite*[5].

Arguments are evaluated using *any* semantics which is based on the notion of *admissibility* [12]. Note that any result that holds under admissible semantics holds also under any semantics based on it. We thus need to recall admissible semantics but also stable one since some results are shown only under this particular semantics.

Definition 4 (Semantics) *Let $(\mathtt{Arg}(\Sigma), \mathcal{R})$ be an AS and $\mathcal{E} \subseteq \mathtt{Arg}(\Sigma)$ and $a \in \mathtt{Arg}(\Sigma)$.*

- *$\mathcal{E}$ is* conflict-free *iff $\nexists\, a, b \in \mathcal{E}$ s.t. $a\mathcal{R}b$.*
- *$\mathcal{E}$* defends *a iff $\forall b \in \mathtt{Arg}(\Sigma)$ s.t. $b\mathcal{R}a$, $\exists c \in \mathcal{E}$ s.t. $c\mathcal{R}b$.*
- *$\mathcal{E}$ is an* admissible extension *iff $\mathcal{E}$ is conflict-free and $\mathcal{E}$ defends any b s.t. $b \in \mathcal{E}$.*
- *$\mathcal{E}$ is a* stable extension *iff $\mathcal{E}$ is conflict-free and for all $b \in \mathtt{Arg}(\Sigma) \setminus \mathcal{E}$, $\exists c \in \mathcal{E}$ s.t. $c\mathcal{R}b$.*

Let $\mathtt{Ext}(\mathcal{T})$ denote the set of all extensions of $\mathcal{T}$ under a given semantics that is based on admissibility, for instance grounded, stable, preferred, etc (see [12] for definitions).

Let us now characterize the conclusions that may be drawn from Σ by an argumentation system. The idea is to infer x from Σ iff it is the conclusion of an argument in each extension.

[3] Generally, the support X is minimal (for set $\subseteq$). In this paper, we do not need to make this assumption.

[4] A set $C \subseteq \Sigma$ is a *minimal conflict* of Σ iff i) C is inconsistent, and ii) $\forall x \in C$, $C \setminus \{x\}$ is consistent.

[5] An AS is *finite* iff each argument is attacked by a finite number of arguments. It is *infinite* otherwise.

Definition 5 (Output) *Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an AS over a knowledge base Σ. For $x \in \mathcal{L}$, $\Sigma \mid\!\sim x$ iff $\forall \mathcal{E} \in \mathtt{Ext}(\mathcal{T})$, $\exists a \in \mathcal{E}$ s.t. $\mathtt{Conc}(a) = x$. $\mathtt{Output}(\mathcal{T})$ = $\{x \in \mathcal{L} \mid \Sigma \mid\!\sim x\}$.*

It is easy to check that the set of outputs coincides with the set of common conclusions of the extensions.

Property 1 *Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an AS over a knowledge base Σ. It holds that $\mathtt{Output}(\mathcal{T}) = \bigcap \mathtt{Concs}(\mathcal{E}_i)$ with $\mathcal{E}_i \in \mathtt{Ext}(\mathcal{T})$.*

It is also obvious that the outputs of an AS are consequences of Σ under CN.

Property 2 *Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an AS over a knowledge base Σ. It holds that $\mathtt{Output}(\mathcal{T}) \subseteq \mathsf{CN}(\Sigma)$.*

It is worth mentioning that an argumentation system starts with a monotonic logic $(\mathcal{L}, \mathsf{CN})$ and defines a *non monotonic logic* $(\mathcal{L}, \mid\!\sim)$. The non monotonicity of $\mid\!\sim$ is obviously due to the status of arguments. An argument may be accepted under a given semantics and becomes rejected when new arguments are received.

3 Postulates for Argumentation Systems

The first rationality postulate that an argumentation system should satisfy concerns the closure of its output. The basic idea is that the conclusions of a formalism should be "complete". A user should not perform on her own some extra reasoning to derive statements that the formalism apparently "forgot" to entail. In [10], closure is defined for rule-based argumentation systems. In what follows, we extend this postulate to systems that are grounded on any Tarskian logic. The idea is to define closure using the consequence operator CN.

Postulate 1 (Closure under CN) *Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an AS over a knowledge base Σ. $\mathcal{T}$ satisfies* closure *i.e. for all $\mathcal{E} \in \mathtt{Ext}(\mathcal{T})$, $\mathtt{Concs}(\mathcal{E}) = \mathsf{CN}(\mathtt{Concs}(\mathcal{E}))$.*

In [10], closure is imposed both on the extensions of an AS and on its output set. The next result shows that the closure of the output set does not deserve to be a separate postulate since it follows immediately from the closure of extensions.

Proposition 1 *Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ. If $\mathcal{T}$ satisfies closure, then $\mathtt{Output}(\mathcal{T})$ = $\mathsf{CN}(\mathtt{Output}(\mathcal{T}))$.*

The second rationality postulate concerns *sub-arguments*. An argument may have one or several sub-arguments, reflecting the different premises on which it is based. Thus, the acceptance of an argument should imply also the acceptance of all its sub-parts. Let us illustrate the importance of this postulate on the following example.

Example 1 *Assume an AS $\mathcal{T}$ built on a propositional knowledge base. Assume also that $\mathtt{Ext}(\mathcal{T}) = \{\mathcal{E}\}$ such that $\mathcal{E} = \{(\{p, p \to \neg f\}, \neg f)\}$, where p stands for penguin and f for fly. This means that the two arguments $(\{p\}, p)$ and $(\{p \to \neg f\}, p \to \neg f)$ are rejected (since they do not belong to $\mathcal{E}$). Thus, the unique accepted argument is grounded on two formulas which are both rejected. It seems counter-intuitive to accept the argument.*

Postulate 2 (Closure under sub-arguments) *Let $\mathcal{T}$ = $(\mathtt{Args}(\Sigma)$, $\mathcal{R})$ be an AS over a knowledge base Σ. $\mathcal{T}$ is* closed under sub-arguments *i.e. for all $\mathcal{E} \in \mathtt{Ext}(\mathcal{T})$, if $a \in \mathcal{E}$, then $\mathtt{Sub}(a) \subseteq \mathcal{E}$.*

It is easy to check that closure under sub-arguments is equivalent to closure under super-arguments. The latter means that if an argument is excluded from an extension, then all arguments built on it (its super-arguments) should also be excluded from that extension.

Property 3 *Let $\mathcal{T} = (\mathtt{Args}(\Sigma), \mathcal{R})$ be an AS over a knowledge base Σ. $\mathcal{T}$ is closed under sub-arguments iff $\forall \mathcal{E} \in \mathtt{Ext}(\mathcal{T})$ if $a \notin \mathcal{E}$, then $\forall b \in \mathtt{Args}(\Sigma)$ s.t. $a \in \mathtt{Sub}(b)$, $b \notin \mathcal{E}$.*

Another interesting property of this postulate is the following.

Property 4 *Let $\mathcal{T} = (\mathtt{Args}(\Sigma), \mathcal{R})$ be an AS over a knowledge base Σ s.t. $\mathcal{T}$ is closed under sub-arguments. $\forall \mathcal{E} \in \mathtt{Ext}(\mathcal{T})$, it holds that:*

- *For all $x \in \mathtt{Base}(\mathcal{E})$, $(\{x\}, x) \in \mathcal{E}$*
- $\mathtt{Base}(\mathcal{E}) \subseteq \mathtt{Concs}(\mathcal{E})$

The next result characterizes the extensions of argumentation systems that are closed under both CN and sub-arguments.

Property 5 *Let $\mathcal{T} = (\mathtt{Args}(\Sigma), \mathcal{R})$ be an AS over a knowledge base Σ. If $\mathcal{T}$ is closed under sub-arguments and under CN, then for all $\mathcal{E} \in \mathtt{Ext}(\mathcal{T})$, $\mathtt{Concs}(\mathcal{E}) = \mathsf{CN}(\mathtt{Base}(\mathcal{E}))$.*

The third rationality postulate concerns the *consistency* of the results. This is the minimum that can be required from a reasoning system. The following postulate generalizes the 'direct consistency postulate' which was proposed for rule-based argumentation systems in [10]. Indeed, we define its counterpart under any Tarskian logic.

Postulate 3 (Consistency) *Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an AS over a knowledge base Σ. $\mathcal{T}$ satisfies* consistency *i.e. for all $\mathcal{E} \in \mathtt{Ext}(\mathcal{T})$, $\mathtt{Concs}(\mathcal{E})$ is consistent.*

As obvious as it may appear, this postulate is violated by some existing argumentation systems like the ASPIC+ system [16]. Let us consider the following example:

Example 2 *Assume that $\mathcal{R} = \{\Rightarrow x, \Rightarrow \neg x \vee y, \Rightarrow \neg y\}$, and that all the other bases defined in [16] are empty. Only three arguments can be built: $A_1 : (\{\Rightarrow x\}, x)$, $A_2 : (\{\Rightarrow \neg x \vee y\}, \neg x \vee y)$, $A_3 : (\{\Rightarrow \neg y\}, \neg y)$. It can be checked that the three arguments are not attacking each other using the attack relation defined in [16]. Thus, the set $\{A_1, A_2, A_3\}$ is an admissible extension. Consequently, the inconsistent set $\{x, \neg x \vee y, \neg y\}$ is the output of the system!*

As for closure, in [10] a postulate imposing the consistency of the output is defined. We show next that such postulate is not necessary since an AS that satisfies Postulate 3, has a consistent output.

Proposition 2 *If $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ satisfies consistency, then the set $\mathtt{Output}(\mathcal{T})$ is consistent.*

In [10], it was shown that some rule-based argumentation systems violate the postulate of *indirect* consistency. Recall that indirect consistency means that the closure (under strict rules) of the conclusions of each extension is consistent. When this postulate is violated, undesirable conclusions may be inferred. We show next that in the case of Tarski's logics, (direct) consistency coincides with indirect consistency. Thus, this latter does not deserve to be a postulate per se.

Proposition 3 *Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an AS over a knowledge base Σ. $\mathcal{T}$ satisfies* consistency *iff for all $\mathcal{E} \in \mathtt{Ext}(\mathcal{T})$, $\mathsf{CN}(\mathtt{Concs}(\mathcal{E}))$ is consistent.*

Until now, we revisited and extended the postulates proposed by Caminada and Amgoud [10]. We showed that three of them (the closure of the output set, the consistency of the output set and indirect consistency) might not be considered as postulates since they follow naturally from more fundamental ones. The question now is: what about the strong version of consistency that is proposed by Amgoud and Besnard [2]? Should it be considered as a postulate or not? Recall that this postulate ensures that for each extension $\mathcal{E}$ of an AS, $\mathtt{Base}(\mathcal{E})$ should be consistent.

Strong Consistency: Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an AS over a knowledge base Σ. $\mathcal{T}$ satisfies *strong consistency* i.e. for all $\mathcal{E} \in \mathtt{Ext}(\mathcal{T})$, $\mathtt{Base}(\mathcal{E})$ is consistent.

This postulate is certainly stronger than Postulate 3.

Proposition 4 *If an AS satisfies strong consistency, then it also satisfies consistency.*

We show next that strong consistency does not deserve to be a postulate per se as it follows from the basic ones, namely consistency and closure under sub-arguments. It is worth mentioning that this result is *very general* as it holds under any semantics, any attack relation and any Tarskian logic.

Proposition 5 *If an AS satisfies consistency and closure under sub-arguments, then it also satisfies strong consistency.*

Two important features of an axiomatic approach is that the postulates should be *independent* and *compatible*, i.e., they may be satisfied together. Hopefully, our three postulates are *independent*. Indeed, the consistency postulate is clearly *independent* from the two others. The following example shows that the two postulates on closure are independent as well.

Example 3 *Assume that $(\mathcal{L}, \mathsf{CN})$ is propositional logic, $\mathcal{T}$ is an AS with a unique extension $\mathcal{E} = \{a, b\}$, $\mathtt{Sub}(a) = \{a\}$, and $\mathtt{Sub}(b) = \{a, b\}$. Thus, $\mathcal{T}$ is closed under sub-arguments. Assume that $\mathtt{Concs}(\mathcal{E}) = \{x, y\}$, then $\mathcal{T}$ violates closure under CN. Assume another AS $\mathcal{T}'$ with a unique extension $\mathcal{E} = \{a, a_1, a_2, ...\}$ where $\mathtt{Conc}(a) = x$ and $\forall a_i$, $\mathtt{Conc}(a_i) = x_i$ with $x_i \in \mathsf{CN}(\{x\})$. Thus, $\mathcal{T}'$ satisfies closure under CN. Assume that $\mathtt{Sub}(a) = \{a, b\}$, then $\mathcal{T}'$ violates closure under sub-arguments.*

The three postulates are also *compatible* as witnessed by the argumentation system studied in [11]. This system is grounded on propositional logic (an instance of Tarski's logic) and uses the assumption attack relation defined in [13]. It was shown that the system satisfies strong consistency under stable semantics. Thus, consistency is also ensured. Besides, each stable extension is closed in terms of arguments ($\mathcal{E} = \mathtt{Arg}(\mathtt{Base}(\mathcal{E}))$), so the system is closed under sub-arguments. Finally, it is easy to check that in this particular system, closure under the consequence operator follows from consistency and closure under sub-arguments.

4 On the Violation of Consistency Postulate

This section studies three properties of attack relations that may lead to the violation of the consistency postulate. The first one concerns the *origin* of the relation. We show that an attack relation should be grounded on inconsistency.

Definition 6 (Conflict-dependent) *An attack relation $\mathcal{R}$ is* conflict-dependent *iff $\forall a, b \in \mathtt{Arg}(\Sigma)$, if $a\mathcal{R}b$ then $\mathtt{Supp}(a) \cup \mathtt{Supp}(b)$ is inconsistent.*

Note that all the attack relations that are used in existing structured argumentation systems are conflict-dependent (see [14] for a summary of those relations). It is very natural that inconsistency would be the origin of the attack relation.

Example 4 *Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an AS built over the propositional knowledge base $\Sigma = \{b, p\}$ where b stands for "Tweety is a bird" and p for "Tweety is a penguin". Assume that $\mathcal{R} = \{(x, y) \mid \mathtt{Supp}(x) \neq \mathtt{Supp}(y)\}$. Note that $\mathcal{R}$ is not conflict-dependent. It is easy to check that $b, p \notin \mathtt{Output}(\mathcal{T})$. This outcome is certainly not intuitive.*

In [2], it was shown that strong consistency is violated by argumentation systems that use a *symmetric* attack relation. One may think that this result is true only when considering the *strong* version of consistency. Unfortunately, it is even true for the weaker version. Indeed, we show that when the attack relation is symmetric, Postulate 3 is violated. Before presenting the result, let us first show some intermediary results. The first one shows that when the knowledge base is a minimal conflict with more than two formulas, then it is possible to build a conflict-free set of arguments.

Lemma 1 *Let $\Sigma = \{x_1, \ldots, x_n\}$ where $n > 2$ and $\mathcal{C}_\Sigma = \{\Sigma\}$. Let $a_1, \ldots, a_n \in \mathtt{Arg}(\Sigma)$ s.t. $\mathtt{Supp}(a_i) = \{x_i\}$. If $\mathcal{R}$ is conflict-dependent, then the set $\mathcal{E} = \{a_1, \ldots, a_n\}$ is conflict-free.*

The previous conflict-free set of arguments defends even its elements when the attack relation is symmetric.

Lemma 2 *Let $\Sigma = \{x_1, \ldots, x_n\}$ where $n > 2$ and $\mathcal{C}_\Sigma = \{\Sigma\}$. Let $a_1, \ldots, a_n \in \mathtt{Arg}(\Sigma)$ s.t. $\mathtt{Supp}(a_i) = \{x_i\}$.*
If $\mathcal{R}$ is conflict-dependent and symmetric, then the set $\mathcal{E} = \{a_1, \ldots, a_n\}$ defends its elements.

From the two lemmas, it follows that the set $\{a_1, \ldots, a_n\}$ is an admissible extension.

Proposition 6 *Let $\Sigma = \{x_1, \ldots, x_n\}$ where $n > 2$ and $\mathcal{C}_\Sigma = \{\Sigma\}$. Let $a_1, \ldots, a_n \in \mathtt{Arg}(\Sigma)$ s.t. $\mathtt{Supp}(a_i) = \{x_i\}$. If $\mathcal{R}$ is conflict-dependent and symmetric, then the set $\mathcal{E} = \{a_1, \ldots, a_n\}$ is an admissible extension.*

The next result shows that the argumentation framework built from the knowledge base $\Sigma = \{x_1, \ldots, x_n\}$ where $n > 2$ and $\mathcal{C}_\Sigma = \{\Sigma\}$ violates consistency.

Proposition 7 *Let $\Sigma = \{x_1, \ldots, x_n\}$ where $n > 2$ and $\mathcal{C}_\Sigma = \{\Sigma\}$. Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over Σ s.t. $\mathcal{R}$ is conflict-dependent and symmetric. $\mathcal{T}$ violates consistency.*

Finally, this result is generalized to any knowledge base containing a ternary or n-ary (with $n > 3$) minimal conflict.

Proposition 8 *Let $\mathcal{C}_\Sigma$ s.t. $\exists C \in \mathcal{C}_\Sigma$ and $|C| > 2$. If $\mathcal{R}$ is conflict-dependent and symmetric, then the system $(\mathtt{Arg}(\Sigma), \mathcal{R})$ violates consistency.*

This result shows a broad class of attack relations that cannot be used in argumentation: the symmetric ones. Thus, relations like rebut or a combination of rebut and any other attack relation would lead to the violation of consistency. Note that this result is conditioned by the existence of n-ary ($n > 2$) minimal conflicts in the knowledge base. The idea is that, due to the binary character of the attack relation, this latter is unable to capture n-ary minimal conflicts.

Another mandatory property that an attack relation should fulfill is that it captures *all* the minimal conflicts of the knowledge base, i.e., each minimal conflict should be captured by at least one attack in $\mathcal{R}$.

Definition 7 (Conflict-exhaustive) *An attack relation $\mathcal{R}$ is* conflict-exhaustive *iff $\forall C \in \mathcal{C}_\Sigma$ s.t. $|C| > 1$, $\exists X \subset C$ s.t. $\exists a, b \in \mathtt{Arg}(\Sigma)$ and $\mathtt{Supp}(a) = X$, $\mathtt{Supp}(b) = C \setminus X$ and either $a\mathcal{R}b$ or $b\mathcal{R}a$.*

Note that an attack relation that is conflict-dependent is not necessarily conflict-exhaustive and vise versa. We show that argumentation systems whose attack relations are not conflict-exhaustive violate consistency. We show progressively this result.

Lemma 3 *Let $\Sigma = \{x_1, \ldots, x_n\}$ where $n > 1$ and $\mathcal{C}_\Sigma = \{\Sigma\}$. Let $a_1, \ldots, a_n \in \mathtt{Arg}(\Sigma)$ s.t. $\mathtt{Supp}(a_i) = \{x_i\}$. If $\mathcal{R}$ is conflict-dependent and not conflict-exhaustive, then the set $\mathcal{E} = \{a_1, \ldots, a_n\}$ is conflict-free.*

Note that symmetric relations are problematic only in presence of ternary or more minimal conflicts, that is a conflict C s.t. $|C| > 2$. However, non conflict-exhaustiveness is fatal even with only binary conflicts. The previous conflict-free set of arguments defends its elements when the attack relation is not conflict-exhaustive.

Lemma 4 *Let $\Sigma = \{x_1, \ldots, x_n\}$ where $n > 1$ and $\mathcal{C}_\Sigma = \{\Sigma\}$. Let $a_1, \ldots, a_n \in \mathtt{Arg}(\Sigma)$ s.t. $\mathtt{Supp}(a_i) = \{x_i\}$. If $\mathcal{R}$ is conflict-dependent and not conflict-exhaustive, then the set $\mathcal{E} = \{a_1, \ldots, a_n\}$ defends its elements.*

From the two lemmas, it follows that the set $\{a_1, \ldots, a_n\}$ is an admissible extension.

Proposition 9 *Let $\Sigma = \{x_1, \ldots, x_n\}$ where $n > 1$ and $\mathcal{C}_\Sigma = \{\Sigma\}$. Let $a_1, \ldots, a_n \in \mathtt{Arg}(\Sigma)$ s.t. $\mathtt{Supp}(a_i) = \{x_i\}$. If $\mathcal{R}$ is conflict-dependent and not conflict-exhaustive, then the set $\mathcal{E} = \{a_1, \ldots, a_n\}$ is an admissible extension.*

The next result shows that the argumentation framework built from the knowledge base $\Sigma = \{x_1, \ldots, x_n\}$ where $n > 1$ and $\mathcal{C}_\Sigma = \{\Sigma\}$ violates consistency.

Proposition 10 *Let $\Sigma = \{x_1, \ldots, x_n\}$ where $n > 1$ and $\mathcal{C}_\Sigma = \{\Sigma\}$. Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over Σ s.t. $\mathcal{R}$ is conflict-dependent and not conflict-exhaustive. $\mathcal{T}$ violates consistency.*

Finally, this result is generalized to any knowledge base containing a binary or more minimal conflict.

Proposition 11 *Let $\mathcal{C}_\Sigma$ s.t. $\exists C \in \mathcal{C}_\Sigma$ and $|C| > 1$. If $\mathcal{R}$ is conflict-dependent and not conflict-exhaustive, then the system $(\mathtt{Arg}(\Sigma), \mathcal{R})$ violates consistency.*

Let us summarize: in order to satisfy consistency, an argumentation system built over a knowledge base under a Tarskian logic should use an attack relation that is conflict-dependent, conflict-exhaustive but not symmetric in case the base contains n-ary (with $n > 2$) minimal conflicts.

5 When are the Postulates Satisfied?

In a previous section, we defined three rationality postulates that *any* argumentation system should satisfy. An important question now is: are there argumentation systems that may satisfy those postulates? If yes, what are the characteristics of those systems? These questions are very ambitious since an argumentation system has three main parameters: the underlying monotonic logic $(\mathcal{L}, \mathsf{CN})$, the attack relation $\mathcal{R}$ and the semantics. In this paper, the three parameters are left unspecified. Thus, getting a complete answer is a real challenge. In this section, we identify one family of argumentation systems that satisfy closure under the consequence operator, three broad families of ASs that satisfy closure under sub-arguments, a broad family of systems that satisfy consistency. The results are general in the sense that they hold under any Tarskian logic, any acceptability semantics, and any attack relation that fulfills the mandatory properties discussed in the previous section.

5.1 Satisfaction of the Closure Postulate

In this section, we identify a class of argumentation systems that satisfy closure under the consequence operator of the underlying logic $(\mathcal{L}, \mathsf{CN})$. We show that an argumentation system that uses an attack relation which captures *all* the minimal conflicts of the knowledge base, and whose extensions contain all the arguments that may be built from the set of formulas appearing in their arguments satisfies closure under CN.

Proposition 12 *Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an AS over a knowledge base Σ such that $\mathcal{R}$ is conflict-exhaustive. If $\forall \mathcal{E} \in \mathtt{Ext}(\mathcal{T})$, $\mathcal{E} = \mathtt{Arg}(\mathtt{Base}(\mathcal{E}))$, then $\mathcal{T}$ is closed under CN.*

It is worth mentioning that the above result holds under *any* acceptability semantics that is based on the notion of conflict-freeness. Thus, it is true for semantics that are not based on admissibility like the ones proposed in [6].

5.2 Satisfaction of the Sub-Arguments Postulate

The satisfaction of Postulate 2 by an argumentation system depends broadly on the properties of its attack relation. We show that when this relation satisfies both rules R_1 and R_2 (see Definition 8), then the system is closed under sub-arguments using admissible semantics (and consequently, under any semantics based on admissibility).

Definition 8 *An attack relation $\mathcal{R}$ satisfies R_1 (resp. R_2) iff $\forall a, b \in \mathtt{Arg}(\Sigma)$ s.t. $\mathtt{Supp}(a) \subseteq \mathtt{Supp}(b)$ and $\forall c \in \mathtt{Arg}(\Sigma)$, it holds $a\mathcal{R}c \Rightarrow b\mathcal{R}c$ (resp. $c\mathcal{R}a \Rightarrow c\mathcal{R}b$).*

The rule R_1 says that if an argument a attacks another argument c, then all the super-arguments of a should also attack c. The second rule says that if an argument a is attacked by an argument c, then all the super-arguments of a should also be attacked by c.

Proposition 13 *Let $\mathcal{T} = (\mathtt{Args}(\Sigma), \mathcal{R})$ be an AS. If $\mathcal{R}$ satisfies R_1 and R_2, then $\mathcal{T}$ satisfies closure under sub-arguments under admissible semantics.*

The next result shows that closure under sub-arguments is less demanding under stable semantics. Indeed, in this case only property R_2 is required for the attack relation.

Proposition 14 *Let $\mathcal{T} = (\mathtt{Args}(\Sigma), \mathcal{R})$ be an AS. If $\mathcal{R}$ satisfies R_2, then $\mathcal{T}$ satisfies closure under sub-arguments under stable semantics.*

The reverse is not necessarily true as shown next.

Example 5 *Let $\mathtt{Arg}(\Sigma) = \{a, b, c, d\}$ be an argumentation system such that $\mathtt{Sub}(b) = \{a, b\}$, $\mathtt{Sub}(a) = \{a\}$, $\mathtt{Sub}(c) = \{c\}$, $\mathtt{Sub}(d) = \{d\}$. Assume also that $c\mathcal{R}a$ and $d\mathcal{R}b$. It is clear that R_2 is violated since c does not attack b. However, the stable extension $\{c, d\}$ is closed wrt sub-arguments.*

The second family of AS that satisfy closure under sub-arguments uses attack relations that are based on and sensible for inconsistency.

Definition 9 (Conflict-sensitive) *An attack relation $\mathcal{R}$ is* conflict-sensitive *iff $\forall a, b \in \mathtt{Arg}(\Sigma)$, if $\mathtt{Supp}(a) \cup \mathtt{Supp}(b)$ is inconsistent, then either $a\mathcal{R}b$ or $b\mathcal{R}a$.*

When the attack relation is conflict-dependent and sensitive, closure under sub-arguments is satisfied.

Proposition 15 *Let $\mathcal{T} = (\mathtt{Args}(\Sigma), \mathcal{R})$ be an AS. If $\mathcal{R}$ is conflict-dependent and conflict-sensitive, then $\mathcal{T}$ satisfies closure under sub-arguments under admissible semantics.*

Notice that the attack relations in the first family of AS are not necessarily based on inconsistency. Finally, we show that argumentation systems whose extensions are closed in terms of arguments enjoy closure under sub-arguments.

Proposition 16 *Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an AS over a knowledge base Σ. If $\forall \mathcal{E} \in \mathtt{Ext}(\mathcal{T})$, $\mathcal{E} = \mathtt{Arg}(\mathtt{Base}(\mathcal{E}))$, then $\mathcal{T}$ is closed under sub-arguments.*

This result is true under *any* acceptability semantics. Indeed, no requirement is needed on the semantics.

5.3 Satisfaction of the Consistency Postulate

In this section, we identify a class of argumentation systems that satisfy consistency. As for closure under sub-arguments, the result depends of the properties of the attack relations. Before that, we start by a result showing a case where consistency coincides with strong consistency.

Proposition 17 *Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system. If $\forall \mathcal{E} \in \mathtt{Ext}(\mathcal{T})$, $\mathcal{E} = \mathtt{Arg}(\mathtt{Base}(\mathcal{E}))$, then $\mathcal{T}$ satisfies consistency implies $\mathcal{T}$ satisfies strong consistency.*

We now show that a system that uses an attack relation which captures *all* the minimal conflicts of the knowledge base and whose extensions contain all the arguments that may be built from the set of formulas appearing in their arguments, satisfies consistency.

Proposition 18 *Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an AS over a knowledge base Σ s.t. $\mathcal{R}$ is conflict-exhaustive. If $\forall \mathcal{E} \in \mathtt{Ext}(\mathcal{T})$, $\mathcal{E} = \mathtt{Arg}(\mathtt{Base}(\mathcal{E}))$, then $\mathcal{T}$ satisfies consistency.*

This result is true under *any* acceptability semantics provided that it is based on the notion of conflict-freeness. Due to Proposition 17, this class of argumentation systems satisfies also strong consistency.

Property 6 *Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an AS over a knowledge base Σ s.t. $\mathcal{R}$ is conflict-exhaustive. If $\forall \mathcal{E} \in \mathtt{Ext}(\mathcal{T})$, $\mathcal{E} = \mathtt{Arg}(\mathtt{Base}(\mathcal{E}))$, then $\mathcal{T}$ satisfies strong consistency.*

This result is very general since, as we already said, the requirement on the attack relation is very natural and even satisfied by all the existing attack relations (see [14] for a review of those relations).

6 Postulates for Weighted Argumentation Systems

Since early nineties, before even the acceptability semantics of Dung [12], arguments were assumed to have different strengths. To the best of my knowledge, the first work on preference-based argumentation systems is the one by Simari and Loui [18]. In that paper, arguments are built from a propositional knowledge base, and the ones that are based on specific information are assumed stronger than those built from general rules. In [8], arguments are built from a possibilistic knowledge base, and are compared following the weakest link principle. The idea is that an argument is better than another one if the weakest formula used in the former is more certain than the weakest formula in the latter. Besides, there is a consensus in the literature on the fact that the strengths of arguments should be taken into account in the evaluation of arguments (e.g. [17, 18]).

The first *abstract* preference-based argumentation framework was proposed in [4]. It takes as input a set of arguments, an attack relation, and a preference relation between arguments which is abstract and can be instantiated in different ways. This proposal was refined in [7] and generalized in [15] in order to reason even about preferences. Thus, arguments may support preferences about arguments. The basic idea behind these frameworks is to ignore an attack if the attacked argument is stronger than its attacker. Dung's semantics are applied on the remaining attacks. In [5], it was shown that these frameworks do not guarantee conflict-free extensions. As a consequence, their instantiations may violate the rationality postulate on consistency. Assume an argumentation system with $\mathcal{E} = \{a, b\}$ as its admissible extension and such that $a\mathcal{R}b$. Since the attack relation should be conflict-dependent, thus $\mathtt{Supp}(a) \cup \mathtt{Supp}(b)$ is certainly inconsistent. From Property 4, if the argumentation system is closed under sub-arguments, then $\mathtt{Supp}(a) \cup \mathtt{Supp}(b) \subseteq \mathtt{Concs}(\mathcal{E})$ meaning that the set of conclusions of $\mathcal{E}$ is inconsistent.

A new approach for preference-based argumentation was proposed in [5]. It takes into account preferences at the semantics level rather than the attack level. The idea is to extend existing acceptability semantics with preferences. In case preferences are not available or do not conflict with the attacks, the extensions of the new semantics coincide with those of the basic ones. This approach computes extensions which are conflict-free. Instantiations of the abstract framework proposed in [5] should satisfy the rationality postulates discussed in the present paper.

7 Conclusion

In this paper we tackled the important problem of defining rational logic-based argumentation systems. We focused on defining postulates that such systems should verify. For that purpose, we revisited and extended the two existing works on the topic [2, 10]. Our contributions are the following:
1) We discussed the existing postulates in the literature, and showed that some of them do not deserve to be postulates per se since they follow from more fundamental ones. This is particularly the case for: strong consistency postulate proposed in [2], output consistency, output closure and indirect consistency that are proposed in [10].
2) We defined three *independent* and *compatible* postulates under *any* Tarskian logic: closure under consequence operator, closure under sub-arguments, and consistency. Recall that two of these postulates were *only* defined under rule-based logics.
3) We provided two families of AS that satisfy closure under sub-arguments, one family of AS that satisfy consistency, and finally two broad families of AS that violate consistency. The results are very general since they hold under any Tarskian logic, any semantics and any attack relation which satisfies some mandatory properties.
4) We discussed the importance of the proposed postulates in preference-based argumentation frameworks.
This work provides guidelines for instantiating Dung's framework as well as its extensions with preferences. It defines the properties that should be ensured. It can also be used for evaluating existing systems. For instance, instantiating Dung's system with canonical undercut [9] as attack relation is certainly a bad choice since the resulting system will violate consistency. Similarly, the ASPIC+ system proposed in [16] violate both consistency and closure under CN (see [1]). In [14] some examples of systems that satisfy consistency are provided. Those systems are built on propositional logic and use specified attack relations.

A lot of work still needs to be done. Our aim is to have a representation theorem that characterizes all the systems that satisfy the three postulates. However, since a system has too many parameters (underlying logic, attack relation, semantics), this objective seems not reachable. Consequently, we will investigate more classes of systems that satisfy the postulates. Another future work consists of investigating more rationality postulates.

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Appendix

Lemma 5 *Let $C \in \mathcal{C}_\Sigma$. For all $X \subset C$, if $X \neq \emptyset$, then $\exists x_1 \in \mathsf{CN}(X)$ and $\exists x_2 \in \mathsf{CN}(C \setminus X)$ such that the set $\{x_1, x_2\}$ is inconsistent.*

Proof Let C be a minimal conflict. Consider $X \subset C$ such that $X \neq \emptyset$. We prove the property by induction, after we first take care to show that X is finite. By Tarski's requirements, there exists $x_0 \in \mathcal{L}$ s.t. $\mathsf{CN}(\{x_0\}) = \mathcal{L}$. Since C is a conflict, $\mathsf{CN}(C) = \mathsf{CN}(\{x_0\})$. As a consequence, $x_0 \in \mathsf{CN}(C)$. However, $\mathsf{CN}(C) = \bigcup_{C' \subseteq_f C} \mathsf{CN}(C')$ by Tarski's requirements. Thus, $x_0 \in \mathsf{CN}(C)$ means that there exists $C' \subseteq_f C$ s.t. $x_0 \in \mathsf{CN}(C')$. This says that C' is a conflict. Since C is a minimal conflict, $C = C'$ and it follows that C is finite. Of course, so is X: Let us write $X = \{x_1, \ldots, x_n\}$. *Base step:* $n = 1$. Taking x to be x_1 is enough. *Induction step:* Assume the lemma is true up to rank $n-1$. As CN is a closure operator, $\mathsf{CN}(\{x_1, \ldots, x_n\}) = \mathsf{CN}(\mathsf{CN}(\{x_1, \ldots, x_{n-1}\}) \cup \{x_n\})$. The induction hypothesis entails $\exists x \in \mathcal{L}$ s.t. $\mathsf{CN}(\mathsf{CN}(\{x_1, \ldots, x_{n-1}\}) \cup \{x_n\}) = \mathsf{CN}(\mathsf{CN}(\{x\}) \cup \{x_n\})$. Then, $\mathsf{CN}(\{x_1, \ldots, x_n\}) = \mathsf{CN}(\{x, x_n\})$. As $\mathsf{CN}(\{x, x_n\}) \neq \mathsf{CN}(\{x_n\})$ and $\mathsf{CN}(\{x, x_n\}) \neq \mathsf{CN}(\{x\})$ (otherwise C cannot be minimal), there exists $y \in \mathcal{L}$ s.t. $\mathsf{CN}(\{x, x_n\}) = \mathsf{CN}(\{y\})$ because $(\mathcal{L}, \mathsf{CN})$ is adjunctive. Since $\mathsf{CN}(\{x_1, \ldots, x_n\}) = \mathsf{CN}(\{x, x_n\})$ was just proved, it follows that $\mathsf{CN}(\{y\}) = \mathsf{CN}(\{x_1, \ldots, x_n\})$.

Take $X_1 = X$ and $X_2 = C \setminus X_1$. Since X is a non-empty proper subset of C, so are both X_1 and X_2. Then, the first bullet of this property can be applied to X_1 and X_2. As a result, $\exists x_1 \in \mathcal{L}$ s.t. $\mathsf{CN}(\{x_1\}) = \mathsf{CN}(X_1)$ and $\exists x_2 \in \mathcal{L}$ s.t. $\mathsf{CN}(\{x_2\}) = \mathsf{CN}(X_2)$. The expansion axiom gives $\{x_1\} \subseteq \mathsf{CN}(\{x_1\})$ and $\{x_2\} \subseteq \mathsf{CN}(\{x_2\})$. Thus, $x_1 \in \mathsf{CN}(X_1)$ and $x_2 \in \mathsf{CN}(X_2)$. Using the expansion axiom again, $X_1 \subseteq \mathsf{CN}(X_1)$ and $X_2 \subseteq \mathsf{CN}(X_2)$. Thus, $X_1 \cup X_2 \subseteq \mathsf{CN}(X_1) \cup \mathsf{CN}(X_2) = \mathsf{CN}(\{x_1\}) \cup \mathsf{CN}(\{x_2\})$. It follows that $C \subseteq \mathsf{CN}(\{x_1\}) \cup \mathsf{CN}(\{x_2\})$. Using Property 1 in [2], $\mathsf{CN}(\{x_1\}) \cup \mathsf{CN}(\{x_2\}) \subseteq \mathsf{CN}(\{x_1, x_2\})$, thus $C \subseteq \mathsf{CN}(\{x_1, x_2\})$. Since C is inconsistent, Property 2 in [2] gives that $\mathsf{CN}(\{x_1, x_2\})$ is inconsistent as well. By the definition of inconsistency, it follows that $\mathsf{CN}(\mathsf{CN}(\{x_1, x_2\})) = \mathcal{L}$. Applying the idempotence axiom, $\mathsf{CN}(\{x_1, x_2\}) = \mathcal{L}$, thus the set $\{x_1, x_2\}$ is inconsistent. ■

Proof of Property 1. Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ.
1) Let $x \in \mathtt{Output}(\mathcal{T})$. Thus, for all $\mathcal{E} \in \mathtt{Ext}(\mathcal{T})$, $\exists a \in \mathcal{E}$ s.t. $\mathtt{Conc}(a) = x$. It follows that $x \in \mathtt{Concs}(\mathcal{E}_i)$, $\forall \mathcal{E}_i \in \mathtt{Ext}(\mathcal{T})$ and hence $x \in \cap\mathtt{Concs}(\mathcal{E}_i)$.
2) Assume that $x \in \cap\mathtt{Concs}(\mathcal{E}_i)$ with $\mathcal{E}_i \in \mathtt{Ext}(\mathcal{T})$. Thus, $\forall \mathcal{E}_i$, $\exists a_i \in \mathcal{E}_i$ s.t. $\mathtt{Conc}(a_i) = x$. Consequently, $x \in \mathtt{Output}(\mathcal{T})$. ■

Proof of Property 2. Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system over a knowledge base Σ. Assume that $x \in \mathtt{Output}(\mathcal{T})$. Thus, from Definition 5, $\exists a \in \mathtt{Arg}(\Sigma)$ such that $\mathtt{Conc}(a) = x$. Since $a \in \mathtt{Arg}(\Sigma)$, then from Definition 2, $\mathtt{Supp}(a) \subseteq \Sigma$ and $x \in \mathsf{CN}(\mathtt{Supp}(a))$. By monotonicity of CN, it follows that $\mathsf{CN}(\mathtt{Supp}(a)) \subseteq \mathsf{CN}(\Sigma)$. Consequently, $x \in \mathsf{CN}(\Sigma)$. ■

Proof of Property 3. Let $\mathcal{T} = (\mathtt{Args}(\Sigma), \mathcal{R})$ be an argumentation system. Let $\mathcal{E}$ be one of its extensions under a given semantics. Assume that $\mathcal{T}$ is closed under sub-arguments and that $b \in \mathtt{Args}(\Sigma)$ but $b \notin \mathcal{E}$. Assume $c \in \mathtt{Args}(\Sigma)$ s.t. $b \in \mathtt{Sub}(c)$ and $c \in \mathcal{E}$. Since $\mathcal{T}$ is closed under sub-arguments, then b would be in $\mathcal{E}$. Contradiction.

Assume now that if $a \notin \mathcal{E}$, then $\forall b \in \mathtt{Args}(\Sigma)$ s.t. $a \in \mathtt{Sub}(b)$, $b \notin \mathcal{E}$. Let $a \in \mathcal{E}$ and assume that $b \in \mathtt{Sub}(a)$ and $b \notin \mathcal{E}$. From the previous property, a should not be in $\mathcal{E}$. ■

Proof of Property 4. Let $\mathcal{T} = (\mathtt{Args}(\Sigma), \mathcal{R})$ be an argumentation system that is closed under sub-arguments. Let $\mathcal{E}$ be one of its extensions under a given semantics and $x \in \mathtt{Base}(\mathcal{E})$. Thus, $\exists a \in \mathcal{E}$ such that $x \in \mathtt{Supp}(a)$. Since $\mathtt{Supp}(a)$ is consistent (by definition of an argument), then the set $\{x\}$ is consistent (from Property 2 in [2]). Thus, the pair $(\{x\}, x)$ is an argument. Moreover, $(\{x\}, x) \in \mathtt{Sub}(a)$. Since $\mathcal{T}$ is closed under sub-arguments, then $(\{x\}, x) \in \mathcal{E}$. ■

Proof of Property 5. Assume that $\mathcal{T} = (\mathtt{Args}(\Sigma), \mathcal{R})$ is closed under sub-arguments and under CN. From Property 4, since $\mathcal{T}$ is closed under sub-arguments, then it follows that $\mathtt{Base}(\mathcal{E}) \subseteq \mathtt{Concs}(\mathcal{E})$. By monotonicity of CN, we get $\mathsf{CN}(\mathtt{Base}(\mathcal{E})) \subseteq \mathsf{CN}(\mathtt{Concs}(\mathcal{E}))$. Since $\mathcal{T}$ is closed under CN, then $\mathsf{CN}(\mathtt{Base}(\mathcal{E})) \subseteq \mathtt{Concs}(\mathcal{E})$.

Besides, by definition of $\mathtt{Concs}(\mathcal{E})$, $\mathtt{Concs}(\mathcal{E}) \subseteq \bigcup \mathsf{CN}(\mathtt{Supp}(a_i))$ with $a_i \in \mathcal{E}$. From Property 1 in [2], it follows that $\mathtt{Concs}(\mathcal{E}) \subseteq \mathsf{CN}(\bigcup \mathtt{Supp}(a_i))$, thus $\mathtt{Concs}(\mathcal{E}) \subseteq \mathsf{CN}(\mathtt{Base}(\mathcal{E}))$. ■

Proof of Proposition 1 Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an AS over a knowledge base Σ. Assume that $\mathcal{T}$ satisfies closure. From Expansion axiom, it follows that $\mathtt{Output}(\mathcal{T}) \subseteq \mathsf{CN}(\mathtt{Output}(\mathcal{T}))$. Assume now that $x \in \mathsf{CN}(\mathtt{Output}(\mathcal{T}))$. Since CN satisfies finiteness, then there exists a finite number of formulas $x_1, \ldots, x_n \in \mathcal{L}$ such that $x_1, \ldots, x_n \in \mathtt{Output}(\mathcal{T})$ and $x \in \mathsf{CN}(\{x_1, \ldots, x_n\})$. From Property 1, $x_1, \ldots, x_n \in \cap\mathtt{Concs}(\mathcal{E}_i)$ where $\mathcal{E}_i \in \mathtt{Ext}(\mathcal{T})$. From monotonicity of CN, it holds that $\mathsf{CN}(\{x_1, \ldots, x_n\}) \subseteq \mathsf{CN}(\cap\mathtt{Concs}(\mathcal{E}_i))$. It holds also that $x \in \mathsf{CN}(\mathtt{Concs}(\mathcal{E}_1)) \cap \ldots \cap \mathsf{CN}(\mathtt{Concs}(\mathcal{E}_n))$. Since $\mathcal{T}$ satisfies closure, then for each $\mathcal{E}_i$ it holds that $\mathsf{CN}(\mathtt{Concs}(\mathcal{E}_i)) = \mathtt{Concs}(\mathcal{E}_i)$. Thus, $x \in \mathtt{Concs}(\mathcal{E}_1) \cap \ldots \cap \mathtt{Concs}(\mathcal{E}_n)$. From Property 1, it holds that $x \in \mathtt{Output}(\mathcal{T})$. ■

Proof of Proposition 2. Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an AS based on a knowledge base Σ. Assume that $\mathcal{T}$ satisfies consistency. Thus, $\forall \mathcal{E}_i \in \mathtt{Ext}(\mathcal{T})$, $\mathtt{Concs}(\mathcal{E}_i)$ is consistent. Let $\mathcal{E}$ be a given extension in the set $\mathtt{Ext}(\mathcal{T})$. Since $\cap\mathtt{Concs}(\mathcal{E}_i) \subseteq \mathtt{Concs}(\mathcal{E})$, then $\cap\mathtt{Concs}(\mathcal{E}_i)$ is consistent as well. Besides, from Property 1, $\mathtt{Output}(\mathcal{T}) = \cap\mathtt{Concs}(\mathcal{E}_i)$. It follows that $\mathtt{Output}(\mathcal{T})$ is consistent. ■

Proof of Proposition 3 Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an AS based on a knowledge base Σ. Assume that $\mathcal{T}$ satisfies consistency. Thus, for all $\mathcal{E} \in \mathtt{Ext}(\mathcal{T})$, $\mathtt{Concs}(\mathcal{E})$ is consistent. Thus, from Property 2 in [2], $\mathsf{CN}(\mathtt{Concs}(\mathcal{E}))$ is consistent.

Assume now that for all $\mathcal{E} \in \mathtt{Ext}(\mathcal{T})$, $\mathsf{CN}(\mathtt{Concs}(\mathcal{E}))$ is consistent. Since by Expansion axiom $\mathtt{Concs}(\mathcal{E}) \subseteq \mathsf{CN}(\mathtt{Concs}(\mathcal{E}))$ then $\mathtt{Concs}(\mathcal{E})$ is consistent. ■

Proof of Proposition 4. Let $\mathcal{T} = (\mathtt{Args}(\Sigma), \mathcal{R})$ be an AS that satisfies strong consistency. Thus, for all $\mathcal{E} \in \mathtt{Ext}(\mathcal{T})$, $\mathtt{Base}(\mathcal{E})$ is consistent. Consequently, $\bigcup_{a_i \in \mathcal{E}} \mathtt{Supp}(a_i)$ is consistent and $\mathsf{CN}(\bigcup_{a_i \in \mathcal{E}} \mathtt{Supp}(a_i))$ is consistent as well (since if X is consistent, then $\mathsf{CN}(X)$ is consistent as well). Besides, for each $a_i \in \mathcal{E}$, $\mathtt{Conc}(a_i) \in \mathsf{CN}(\mathtt{Supp}(a_i))$. Thus, $\mathtt{Concs}(\mathcal{E}) \subseteq \cup\mathsf{CN}(\mathtt{Supp}(a_i))$. It follows that $\mathtt{Concs}(\mathcal{E}) \subseteq \mathsf{CN}(\bigcup_{a_i \in \mathcal{E}} \mathtt{Supp}(a_i))$. Since $\mathsf{CN}(\bigcup_{a_i \in \mathcal{E}} \mathtt{Supp}(a_i))$ is consistent, then its subset $\mathtt{Concs}(\mathcal{E})$ is consistent. ■

Proof of Proposition 5 Let $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ be an AS over a knowledge base Σ. Assume that $\mathcal{T}$ satisfies consistency and closure under sub-arguments. From closure under sub-arguments, it follows that for all $\mathcal{E} \in \texttt{Ext}(\mathcal{T})$, $\texttt{Base}(\mathcal{E}) \subseteq \texttt{Concs}(\mathcal{E})$ (Property 4). Since $\mathcal{T}$ satisfies consistency, then the set $\texttt{Concs}(\mathcal{E})$ is consistent. From Property 2 in [2], it follows that $\texttt{Base}(\mathcal{E})$ is consistent. ■

Proof of Lemma 1 Let $\Sigma = \{x_1, \ldots, x_n\}$ where $n > 2$ and $\mathcal{C}_\Sigma = \{\Sigma\}$, and let $(\texttt{Arg}(\Sigma), \mathcal{R})$ be an AS such that $\mathcal{R}$ is conflict-dependent. Let $a_1, \ldots, a_n \in \texttt{Arg}(\Sigma)$ be such that $\texttt{Supp}(a_i) = \{x_i\}$. Assume that the set $\mathcal{E} = \{a_1, \ldots, a_n\}$ is not conflict-free. Thus, $\exists a_i, a_j \in \mathcal{E}$ such that $a_i \mathcal{R} a_j$. Since $\mathcal{R}$ is conflict-dependent, then $\texttt{Supp}(a_i) \cup \texttt{Supp}(a_j)$ is inconsistent. This is impossible since $|\texttt{Supp}(a_i) \cup \texttt{Supp}(a_j)| < n$ and thus, from the definition of a minimal conflict, $\texttt{Supp}(a_i) \cup \texttt{Supp}(a_j)$ should be consistent. ■

Proof of Lemma 2 Let $\Sigma = \{x_1, \ldots, x_n\}$ where $n > 2$ and $\mathcal{C}_\Sigma = \{\Sigma\}$, and let $(\texttt{Arg}(\Sigma), \mathcal{R})$ be an AS such that $\mathcal{R}$ is conflict-dependent. Let $a_1, \ldots, a_n \in \texttt{Arg}(\Sigma)$ be such that $\texttt{Supp}(a_i) = \{x_i\}$. Assume that the set $\mathcal{E} = \{a_1, \ldots, a_n\}$ does not defend its elements. Thus, $\exists a_i \in \mathcal{E}$ such that $\exists b \in \texttt{Arg}(\Sigma)$ and $b \mathcal{R} a_i$ and $\mathcal{E}$ does not defend a_i. This is impossible since $\mathcal{R}$ is symmetric thus, $a_i \mathcal{R} b$. ■

Proof of Proposition 6 Let $\Sigma = \{x_1, \ldots, x_n\}$ where $n > 2$ and $\mathcal{C}_\Sigma = \{\Sigma\}$, and let $(\texttt{Arg}(\Sigma), \mathcal{R})$ be an AS such that $\mathcal{R}$ is conflict-dependent. Let $a_1, \ldots, a_n \in \texttt{Arg}(\Sigma)$ be such that $\texttt{Supp}(a_i) = \{x_i\}$. From Lemma 1, the set $\mathcal{E} = \{a_1, \ldots, a_n\}$ is conflict-free and from Lemma 2 it defends its elements. Thus, $\mathcal{E} = \{a_1, \ldots, a_n\}$ is an admissible set. ■

Proof of Proposition 7 Let $\Sigma = \{x_1, \ldots, x_n\}$ where $n > 2$ and $\mathcal{C}_\Sigma = \{\Sigma\}$, and let $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ be an AS such that $\mathcal{R}$ is conflict-dependent. Let $a_1, \ldots, a_n \in \texttt{Arg}(\Sigma)$ be such that $\texttt{Supp}(a_i) = \{x_i\}$ and $\texttt{Conc}(a_i) = x_i$. From Proposition 6, the set $\mathcal{E} = \{a_1, \ldots, a_n\}$ is an admissible set. Besides, $\texttt{Concs}(\mathcal{E}) = \{x_1, \ldots, x_n\}$, thus $\mathcal{T}$ violates consistency. ■

Proof of Proposition 8 Let $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ be an AS over a base Σ s.t. $\mathcal{R}$ is both conflict-dependent and symmetric. Consider $C = \{x_1, \ldots, x_n\}$ where $n > 2$ and assume that $C \in \mathcal{C}_\Sigma$. It follows from Proposition 6 that the set $\mathcal{E} = \{a_1, \ldots, a_n\}$, with $\texttt{Supp}(a_i) = \{x_i\}$ and $\texttt{Conc}(a_i) = x_i$, is an admissible extension of $\mathcal{T}$. Moreover, $\texttt{Concs}(\mathcal{E})$ is inconsistent. Thus, $\mathcal{T}$ violates consistency. ■

Proof of Lemma 3 Let $\Sigma = \{x_1, \ldots, x_n\}$ where $n > 1$ and $\mathcal{C}_\Sigma = \{\Sigma\}$, and let $(\texttt{Arg}(\Sigma), \mathcal{R})$ be an AS such that $\mathcal{R}$ is conflict-dependent and not conflict-exhaustive. Let $a_1, \ldots, a_n \in \texttt{Arg}(\Sigma)$ be such that $\texttt{Supp}(a_i) = \{x_i\}$. Assume that the set $\mathcal{E} = \{a_1, \ldots, a_n\}$ is not conflict-free. Thus, $\exists a_i, a_j \in \mathcal{E}$ such that $a_i \mathcal{R} a_j$. Since $\mathcal{R}$ is conflict-dependent, then $\texttt{Supp}(a_i) \cup \texttt{Supp}(a_j)$ is inconsistent. If $n = 2$, then this is impossible since $\mathcal{R}$ is not conflict-exhaustive. If $n > 2$ this is again impossible since $|\texttt{Supp}(a_i) \cup \texttt{Supp}(a_j)| < n$ and thus, from the definition of a minimal conflict, $\texttt{Supp}(a_i) \cup \texttt{Supp}(a_j)$ should be consistent. ■

Proof of Lemma 4 Let $\Sigma = \{x_1, \ldots, x_n\}$ where $n > 1$ and $\mathcal{C}_\Sigma = \{\Sigma\}$, and let $(\texttt{Arg}(\Sigma), \mathcal{R})$ be an AS such that $\mathcal{R}$ is conflict-dependent and not conflict-exhaustive. Let $a_1, \ldots, a_n \in \texttt{Arg}(\Sigma)$ be such that $\texttt{Supp}(a_i) = \{x_i\}$. Assume that the set $\mathcal{E} = \{a_1, \ldots, a_n\}$ does not defend its elements. Thus, $\exists a_i \in \mathcal{E}$ such that $\exists b \in \texttt{Arg}(\Sigma)$ and $b \mathcal{R} a_i$ and $\mathcal{E}$ does not defend a_i. Since $\mathcal{R}$ is conflict-dependent, then $\texttt{Supp}(a_i) \cup \texttt{Supp}(b)$ is inconsistent. Thus, $\texttt{Supp}(a_i) \cup \texttt{Supp}(b) = \Sigma$. Consequently, $\texttt{Supp}(b) = \Sigma \setminus \texttt{Supp}(a_i)$. This is impossible since $\mathcal{R}$ is not conflict-exhaustive. ■

Proof of Proposition 9 Let $\Sigma = \{x_1, \ldots, x_n\}$ where $n > 1$ and $\mathcal{C}_\Sigma = \{\Sigma\}$, and let $(\texttt{Arg}(\Sigma), \mathcal{R})$ be an AS such that $\mathcal{R}$ is conflict-dependent and not conflict-exhaustive. Let $a_1, \ldots, a_n \in \texttt{Arg}(\Sigma)$ be such that $\texttt{Supp}(a_i) = \{x_i\}$. From Lemma 3, the set $\mathcal{E} = \{a_1, \ldots, a_n\}$ is conflict-free and from Lemma 4 it defends its elements. Thus, $\mathcal{E} = \{a_1, \ldots, a_n\}$ is an admissible set. ■

Proof of Proposition 10 Let $\Sigma = \{x_1, \ldots, x_n\}$ where $n > 1$ and $\mathcal{C}_\Sigma = \{\Sigma\}$, and let $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ be an AS such that $\mathcal{R}$ is conflict-dependent and not conflict-exhaustive. Let $a_1, \ldots, a_n \in \texttt{Arg}(\Sigma)$ be such that $\texttt{Supp}(a_i) = \{x_i\}$ and $\texttt{Conc}(a_i) = x_i$. From Proposition 9, the set $\mathcal{E} = \{a_1, \ldots, a_n\}$ is an admissible set. Besides, $\texttt{Concs}(\mathcal{E}) = \{x_1, \ldots, x_n\}$, thus $\mathcal{T}$ violates consistency. ■

Proof of Proposition 11 Let $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ be an AS over a base Σ s.t. $\mathcal{R}$ is conflict-dependent but not conflict-exhaustive. Thus, there exists $C = \{x_1, \ldots, x_n\}$ such that C is not captured by $\mathcal{R}$. It follows from Proposition 6 that the set $\mathcal{E} = \{a_1, \ldots, a_n\}$, with $\texttt{Supp}(a_i) = \{x_i\}$ and $\texttt{Conc}(a_i) = x_i$, is an admissible extension of $\mathcal{T}$. Moreover, from Proposition 10, $\mathcal{E}$ violates consistency. Thus, $\mathcal{T}$ violates extension consistency. ■

Proof of Proposition 12 Let $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ be an AS s.t. $\mathcal{R}$ is conflict-exhaustive and $\forall \mathcal{E} \in \texttt{Ext}(\mathcal{T})$, $\mathcal{E} = \texttt{Arg}(\texttt{Base}(\mathcal{E}))$.
Let $\mathcal{E}$ be an admissible extension and $x \in \mathsf{CN}(\texttt{Concs}(\mathcal{E}))$. Thus, $\exists \{x_1, \ldots, x_n\} \subseteq \texttt{Concs}(\mathcal{E})$ s.t. $x \in \mathsf{CN}(\{x_1, \ldots, x_n\})$. Besides, $\forall x_i$, $\exists a_i \in \mathcal{E}$ s.t. $x_i \in \mathsf{CN}(\texttt{Supp}(a_i))$. Thus, $\{x_1, \ldots, x_n\} \subseteq \bigcup_{i=1,n} \mathsf{CN}(\texttt{Supp}(a_i))$. From Property 1 in [2], $\bigcup_{i=1,n} \mathsf{CN}(\texttt{Supp}(a_i)) \subseteq \mathsf{CN}(\bigcup_{i=1,n} \texttt{Supp}(a_i))$. Then, $\{x_1, \ldots, x_n\} \subseteq \mathsf{CN}(\bigcup_{i=1,n} \texttt{Supp}(a_i))$ and $x \in \mathsf{CN}(\bigcup_{i=1,n} \texttt{Supp}(a_i))$. From Property 6, $\texttt{Base}(\mathcal{E})$ is consistent. Since $\bigcup_{i=1,n} \texttt{Supp}(a_i) \subseteq \texttt{Base}(\mathcal{E})$, then $\bigcup_{i=1,n} \texttt{Supp}(a_i)$ is consistent (see Property 2 in [2]). Consequently, the pair $(\bigcup_{i=1,n} \texttt{Supp}(a_i), x)$ is an argument. Hence, $(\bigcup_{i=1,n} \texttt{Supp}(a_i), x) \in \texttt{Arg}(\texttt{Base}(\mathcal{E}))$ and thus, $(\bigcup_{i=1,n} \texttt{Supp}(a_i), x) \in \mathcal{E}$. It follows that $x \in \texttt{Concs}(\mathcal{E})$. ■

Proof of Proposition 13 Let $\mathcal{T} = (\texttt{Args}(\Sigma), \mathcal{R})$ be an AS such that $\mathcal{R}$ satisfies R_1 and R_2. Let $\mathcal{E}$ be an admissible extension of $\mathcal{T}$. Assume that $\mathcal{E}$ is not closed under sub-arguments. Thus, $\exists a \in \mathcal{E}$ such that $\texttt{Sub}(a) \not\subseteq \mathcal{E}$. This means that $\exists a' \in \texttt{Sub}(a)$ and $a' \notin \mathcal{E}$. Two possibilities hold:

1. $\mathcal{E} \cup \{a'\}$ is conflicting. Thus, $\exists b \in \mathcal{E}$ such that either $a' \mathcal{R} b$ or $b \mathcal{R} a'$ hold. Assume that $a' \mathcal{R} b$. Since $a' \in \texttt{Sub}(a)$ and $\mathcal{R}$ verifies R_1, then $a \mathcal{R} b$. This contradicts the fact that $\mathcal{E}$ is admissible. Assume now that $b \mathcal{R} a'$. Since $\mathcal{R}$ satisfies R_2, then $b \mathcal{R} a$, contradiction.
2. $\mathcal{E}$ does not defend a'. Thus, $\exists b \notin \mathcal{E}$ such that $b \mathcal{R} a'$ and $\nexists c \in \mathcal{E}$ such that $c \mathcal{R} b$. Since $b \mathcal{R} a'$ and $\mathcal{R}$ satisfies R_2, then $b \mathcal{R} a$. Since $a \in \mathcal{E}$ and $\mathcal{E}$ is admissible, this means that $\exists c \in \mathcal{E}$ such that $c \mathcal{R} b$. Contradiction. ■

Proof of Proposition 14 Let $\mathcal{T} = (\texttt{Args}(\Sigma), \mathcal{R})$ be an AS such that $\mathcal{R}$ satisfies R_2. Let $\mathcal{E}$ be a stable extension of $\mathcal{T}$ which is not closed under sub-arguments. Thus, $\exists a \in \mathcal{E}$ such that $\texttt{Sub}(a) \not\subseteq \mathcal{E}$. This means that $\exists a' \in \texttt{Sub}(a)$ and $a' \notin \mathcal{E}$. Then, $\exists b \in \mathcal{E}$ such that $b \mathcal{R} a'$ (according to the definition of a stable extension). Since $\mathcal{R}$ satisfies R_2, then $b \mathcal{R} a$. This contradicts the fact that $\mathcal{E}$ is conflict-free. ■

Proof of Proposition 15 Let $\mathcal{E}$ be an admissible extension of an AS $\mathcal{T} = (\texttt{Args}(\Sigma), \mathcal{R})$. Assume that $\mathcal{R}$ is conflict-dependent and sensitive. Assume also that $\mathcal{E}$ is not closed under sub-arguments. That is,

$\exists a, a' \in \texttt{Arg}(\Sigma)$ s.t. $a' \in \texttt{Sub}(a)$, $a \in \mathcal{E}$ and $a' \notin \mathcal{E}$. Two situations are possible:

1. $\mathcal{E} \cup \{a'\}$ is conflicting meaning that $\exists b \in \mathcal{E}$ s.t. either $a'\mathcal{R}b$ or $b\mathcal{R}a'$. Since $\mathcal{R}$ is conflict-dependent, then $\texttt{Supp}(a') \cup \texttt{Supp}(b)$ is inconsistent. Besides, $a' \in \texttt{Sub}(a)$ thus $\texttt{Supp}(a') \subseteq \texttt{Sub}(a)$. From Property 2 in [2], $\texttt{Supp}(a) \cup \texttt{Supp}(b)$ is inconsistent as well. Since $\mathcal{R}$ is conflict-sensitive, then either $a\mathcal{R}b$ or $b\mathcal{R}a$. This contradicts the fact $\mathcal{E}$ is conflict-free.
2. $\mathcal{E}$ does not defend a'. Thus, $\exists b \in \texttt{Arg}(\Sigma)$ s.t. $b\mathcal{R}a'$. Since $\mathcal{R}$ is conflict-dependent, then $\texttt{Supp}(a') \cup \texttt{Supp}(b)$ is inconsistent. Besides, $a' \in \texttt{Sub}(a)$ then $\texttt{Supp}(a') \subseteq \texttt{Sub}(a)$. Thus, $\texttt{Supp}(a) \cup \texttt{Supp}(b)$ is inconsistent as well. Since $\mathcal{R}$ is conflict-sensitive, then either $a\mathcal{R}b$ or $b\mathcal{R}a$. Assume that $a\mathcal{R}b$, thus a defends a' which contradicts the fact that $\mathcal{E}$ does not defend a'. Assume now that $b\mathcal{R}a$. Since $\mathcal{E}$ is admissible and $a \in \mathcal{E}$, then $\exists c \in \mathcal{E}$ s.t. $c\mathcal{R}b$. Thus, c defends even a', this contradicts again the fact that $\mathcal{E}$ does not defend a'. ■

Proof of Proposition 16 Let $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ be an AS over a knowledge base Σ. Assume that $\forall \mathcal{E} \in \texttt{Ext}(\mathcal{T})$, $\mathcal{E} = \texttt{Arg}(\texttt{Base}(\mathcal{E}))$. Let $\mathcal{E} \in \texttt{Ext}(\mathcal{T})$ and $a \in \mathcal{E}$. Since $a \in \mathcal{E}$, then $\texttt{Supp}(a) \subseteq \texttt{Base}(\mathcal{E})$. Let $b \in \texttt{Sub}(a)$, thus $\texttt{Supp}(b) \subseteq \texttt{Supp}(a)$ and $\texttt{Supp}(b) \subseteq \texttt{Base}(\mathcal{E})$. It follows that $b \in \texttt{Arg}(\texttt{Base}(\mathcal{E}))$. Consequently, $b \in \mathcal{E}$. Then, $\mathcal{T}$ is closed under sub-arguments. ■

Proof of Proposition 17 Let $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ be an AS such that $\forall \mathcal{E} \in \texttt{Ext}(\mathcal{T})$, $\mathcal{E} = \texttt{Arg}(\texttt{Base}(\mathcal{E}))$. Assume that $\mathcal{T}$ violates strong consistency. Thus, there exists an extension $\mathcal{E}$ of $\mathcal{T}$ (under a given semantics) such that $\texttt{Base}(\mathcal{E})$ is inconsistent. Thus, $\exists C \in \mathcal{C}_\Sigma$ such that $C \subseteq \texttt{Base}(\mathcal{E})$. Since $\texttt{Base}(\mathcal{E}) = \bigcup_{a_i \in \mathcal{E}} \texttt{Supp}(a_i)$ and $\texttt{Supp}(a_i)$ is consistent, then $|C| \geq 2$. Thus, $\exists X \subset C$ such that X and $C \setminus X$ are consistent. From Proposition 1 (in [2]), there exist two arguments a and b where $\texttt{Supp}(a) = X$ and $\texttt{Supp}(b) = C \setminus X$. From Lemma 5, $\exists x_1 \in \mathsf{CN}(X)$ and $\exists x_2 \in \mathsf{CN}(C \setminus X)$ such that the set $\{x_1, x_2\}$ in inconsistent. Let $\texttt{Conc}(a) = x_1$ and $\texttt{Conc}(b) = x_2$. Since $a, b \in \texttt{Arg}(\texttt{Base}(\mathcal{E}))$ and that $\mathcal{E} = \texttt{Arg}(\texttt{Base}(\mathcal{E}))$, then $a, b \in \mathcal{E}$. Thus, $\texttt{Concs}(\mathcal{E})$ is inconsistent. ■

Proof of Proposition 18 Let $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ be an AS over a knowledge base Σ s.t. $\mathcal{R}$ is conflict-exhaustive and for each $\mathcal{E} \in \texttt{Ext}(\mathcal{T})$, $\mathcal{E} = \texttt{Arg}(\texttt{Base}(\mathcal{E}))$. Note that in this case, consistency coincides with strong consistency (from Proposition 17).

Let $\mathcal{E}$ be an admissible extension of $\mathcal{T}$ s.t. $\texttt{Base}(\mathcal{E})$ is inconsistent. Thus, $\exists C \in \mathcal{C}_\Sigma$ s.t. $C \subseteq \texttt{Base}(\mathcal{E})$. Since $\texttt{Base}(\mathcal{E}) = \bigcup \texttt{Supp}(a_i)$ ($a_i \in \mathcal{E}$) and $\texttt{Supp}(a_i)$ is consistent (by definition of an argument), then $|C| \geq 2$. Since $\mathcal{R}$ is conflict-exhaustive, then $\exists X \subset C$ s.t. $\exists a, b \in \texttt{Arg}(\Sigma)$ and $\texttt{Supp}(a) = X$, $\texttt{Supp}(b) = C \setminus X$ and either $a\mathcal{R}b$ or $b\mathcal{R}a$. Besides, $\texttt{Supp}(a) \subseteq \texttt{Base}(\mathcal{E})$ (resp. $\texttt{Supp}(b) \subseteq \texttt{Base}(\mathcal{E})$), then $a, b \in \texttt{Arg}(\texttt{Base}(\mathcal{E}))$. Since $\mathcal{E} = \texttt{Arg}(\texttt{Base}(\mathcal{E}))$, then $a, b \in \mathcal{E}$. This means that the extension $\mathcal{E}$ is conflicting. Contradiction. ■