On argumentation-based paraconsistent logics

Leila Amgoud

IRIT – CNRS
118, route de Narbonne
31062, Toulouse Cedex 09
amgoud@irit.fr

Abstract. *Argumentation* is an alternative approach for reasoning with inconsistent information. Starting from a knowledge base (a set of premises) encoded in a logical language, an argumentation-based logic defines *arguments* and *attacks* between them using the consequence operator associated with the language, then uses a *semantics* for evaluating the arguments. The plausible conclusions to be drawn from the knowledge base are those supported by "good" arguments.
In this paper, we discuss two families of such logics: the family of logics that uses extension semantics for the evaluation of arguments, and the one that uses ranking semantics. We discuss the outcomes of both families and compare them.

1 Introduction

An important problem in the management of knowledge-based systems is handling of inconsistency. Inconsistency may be present for mainly three reasons: i) A knowledge base may contain a default rule and strict rules encoding exceptions of the default rule [26]. The two kinds of rules may lead to opposite conclusions. ii) In model-based diagnosis [21], the description of the normal behavior of a system may be conflicting with the observations made on this system. iii) An inconsistent knowledge base may result from the union of several consistent knowledge bases pertaining to the same domain [12]. Moreover, in [18], Gabbay and Hunter claim that inconsistency in a database exists on purpose and may be useful if its presence triggers suitable actions that cope with it. They give the example of overbooking in airline booking systems.

Whatever the source of inconsistency, a paraconsistent logic is needed to deal with it. A paraconsistent logic consists of a language and a consequence operator which returns rational conclusions even from inconsistent sets of formulas.

There has been much work on constructing and investigating such logics. Two families can be distinguished: those that restore consistency (e.g., [10, 26, 27]) and those that tolerate inconsistency and cope with it (e.g., [6, 7, 13, 17]). One important instance of the first family computes the maximal (for set inclusion) consistent subbases of a knowledge base, then chooses the conclusions that follow from all those subbases ([27]). Regarding the second family, a prominent approach considers many-valued interpretations with the crucial particularity that they can be models of even inconsistent premises and thus, can be used to draw conclusions.

Since early nineties, due to its explanatory power, *argumentation* has become a promising approach for handling inconsistency. Like many-valued logics, it accepts inconsistency and copes with it. Starting from a knowledge base encoded in a particular logical language, an argumentation-based logic builds arguments and attack relations between them using a consequence operator associated with the language, then it evaluates the arguments using a semantics. Finally, it draws the conclusions that are supported by good arguments. Attacks generally refer to the inconsistency of the base.

In the argumentation literature, arguments are mainly evaluated using *extension*-based semantics (or extension semantics for short) as introduced by Dung in his seminal paper [15]. Extension semantics are functions transforming any argumentation graph into one or several subsets of arguments, called *extensions*, each of which representing a coherent point of view. Using the extensions, the set of arguments is partitioned into three disjoint categories: i) the arguments which are in all extensions (called *sceptically* accepted), ii) the arguments that are in some but not all extensions (called *credulously* accepted), and ii) the arguments which do not belong to any extension (called *rejected*). Examples of extension semantics are the well-known stable, preferred, complete, grounded, and admissible semantics proposed by Dung in [15], as well as their refinements like the recursive semantics [5] and ideal semantics [14].

More recently, another family of semantics, called *graded semantics*, is emerging. Those semantics do not compute extensions and are based on different principles. For instance, the number of attackers is taken into account while it does not play any role in extension semantics. Thus, the two families of semantics (extension semantics and graded semantics) may not provide the same evaluations of arguments.

Graded semantics assign to every argument a numerical value representing its *strength*. These values allow to rank-order the arguments from the most acceptable to the less acceptable ones. Ranking semantics [1], h-categoriser semantics [8, 25], value-based semantics [11], and game-theoretical semantics [22] are examples of graded semantics.

In [29], the authors presented a very interesting extension of Dung's semantics by introducing the notion of *stratified labellings*. Like graded semantics, the idea is to provide graded assessment of arguments by assigning a degree to each argument. However, the degree does not represent the strength of an argument but rather to what extent the argument is controversial. Thus, the corresponding semantics do not satisfy the mandatory properties of graded semantics. Interestingly enough, stratified labellings have close relationships with ranking functions like Z-ordering [23].

All the above semantics evaluate arguments solely on the basis of attacks and do not take into account the internal structure of arguments. Their input is a plain directed graph whose nodes and arrows represent abstract arguments and attacks.

Our aim in this paper is to discuss and compare the paraconsistent logics built on top of any Tarskian logic ([28]) and induced by each family of semantics. We consider in particular the most popular extension semantics (stable and preferred semantics introduced by Dung in [15]) and a *ranking semantics* defined more recently in [1]. We show that logics based on extension semantics (ALES) return flat conclusions and restore consistency. Furthermore, they generalize to any Tarskian logic the paraconsistent logic defined by Rescher and Manor in [27] on top of propositional logic. The paraconsistent logics based on ranking semantics (ALRS) return ranked conclusions (from the most plausible to the less plausible ones) and tolerate inconsistency. Moreover, they are good candidates for measuring inconsistency in knowledge bases. Finally, we show that ALRS are more discriminating than ALES in that they solve inconsistency while ALES avoid it.

2 Argumentation-based logics

An argumentation-based logic is built on top of a logic. In this paper, we focus on Tarskian logics [28]. According to Tarski, a logic is a set of well-formed *formulae* and a *consequence operator* which returns the set of formulae that follow from another set of formulae. There are no requirements on the connectives used in the language. However, the consequence operator should satisfy some very basic properties.

Definition 1 (Logic) *A logic is a tuple $\langle \mathcal{F}, w, \mathtt{CN} \rangle$ where $\mathcal{F}$ is a set of well-formed formulae, w is a well-order*[1] *on $\mathcal{F}$, $\mathtt{CN}$ is a consequence operator, i.e., a function from $2^{\mathcal{F}}$ to $2^{\mathcal{F}}$ such that for $\Phi \subseteq \mathcal{F}$,*

- $\Phi \subseteq \mathtt{CN}(\Phi)$ *(Expansion)*
- $\mathtt{CN}(\mathtt{CN}(\Phi)) = \mathtt{CN}(\Phi)$ *(Idempotence)*
- $\mathtt{CN}(\Phi) = \bigcup_{\Psi \subseteq_f \Phi} \mathtt{CN}(\Psi)$ *(Compactness)*
- $\mathtt{CN}(\{\phi\}) = \mathcal{F}$ *for some* $\phi \in \mathcal{F}$ *(Absurdity)*
- $\mathtt{CN}(\emptyset) \neq \mathcal{F}$ *(Coherence)*

Notation: $Y \subseteq_f X$ means that Y is a finite subset of X.

The well-ordering w enables to arbitrarily select a *representative* formula among equivalent ones. Its exact definition is not important for the purpose of the paper. Almost all well-known monotonic logics (classical logics, intuitionistic logics, modal logics, etc.) can be viewed as special cases of Tarski's notion of an abstract logic. AI introduced non-monotonic logics, which do not satisfy monotonicity [9].

The next definition introduces the concept of *adjunctive* logic.

Definition 2 (Adjunctiveness) *A logic $\langle \mathcal{F}, w, \mathtt{CN} \rangle$ is* adjunctive *iff for all ϕ and ψ in $\mathcal{F}$, there exists $\alpha \in \mathcal{F}$ such that* $\mathtt{CN}(\{\alpha\}) = \mathtt{CN}(\{\phi, \psi\})$.

Intuitively, an adjunctive logic infers, from the union of two formulas $\{\phi, \psi\}$, some formula(s) that can be inferred neither from ϕ alone nor from ψ alone (except, of course, when ψ ensues from ϕ or vice-versa). In fact, most well-known logics are adjunctive.[2] A logic which is not adjunctive could for instance fail to deny $\phi \vee \psi$ from the premises $\{\neg\phi, \neg\psi\}$.

The notion of *consistency* associated with such logics is defined as follows:

Definition 3 (Consistency) *A set $\Phi \subseteq \mathcal{L}$ is* consistent *wrt a logic $(\mathcal{L}, w, \mathtt{CN})$ iff* $\mathtt{CN}(\Phi) \neq \mathcal{L}$. *It is* inconsistent *otherwise.*

Before introducing the notion of argument, let us first define when pairs of formulas are equivalent.

---

[1] A *well-order* on a set X is a relation with the property that every non-empty subset of X has a least element in this ordering.

[2] Some fragments of well-known logics fail to be adjunctive, e.g., the pure implicational fragment of classical logic as it is negationless, disjunctionless, and, of course, conjunctionless.

Definition 4 (Equivalent formulas) *Let $\langle \mathcal{F}, w, \mathtt{CN} \rangle$ be a logic and $\phi, \psi \in \mathcal{F}$. The formula ϕ is* equivalent *to ψ wrt logic $\langle \mathcal{F}, w, \mathtt{CN} \rangle$, denoted by $\phi \equiv \psi$, iff $\mathtt{CN}(\{\phi\}) = \mathtt{CN}(\{\psi\})$.*

The building block of argumentation-based logics is the notion of argument. An argument is a reason for concluding a formula. Thus, it has two main components: a *support* and a *conclusion*. In what follows, two arguments having the same supports and different yet equivalent conclusions are not distinguished, they are rather seen as the same argument. The reason is that those arguments are redundant and increase uselessly and misleadingly the argumentation graph both from a theoretical and computational point of view.

Definition 5 (Argument) *Let $\langle \mathcal{F}, w, \mathtt{CN} \rangle$ be a logic and $\Phi \subseteq_f \mathcal{F}$. An* argument *built from Φ is a pair (Ψ, ψ) such that:*

- *$\Psi \subseteq \Phi$ and Ψ is consistent,*
- *ψ is the w-smallest element of $\{\psi' \in \mathcal{F} \mid \psi' \equiv \psi\}$ such that $\psi \in \mathtt{CN}(\Psi)$,*
- *$\nexists \Psi' \subset \Psi$ such that $\psi \in \mathtt{CN}(\Psi')$.*

An argument (Ψ, ψ) is a sub-argument *of (Ψ', ψ') iff $\Psi \subseteq \Psi'$.*

Notations: Let $\langle \mathcal{F}, w, \mathtt{CN} \rangle$ be a logic and $\Phi \subseteq_f \mathcal{F}$. $\mathtt{Supp}$ and $\mathtt{Conc}$ denote respectively the *support* Ψ and the *conclusion* ψ of an argument (Ψ, ψ) built from Φ. $\mathtt{Arg}(\Phi)$ denotes the set of all arguments that can be built from Φ by means of Definition 5, $\mathtt{Sub}((\Psi, \psi))$ is a function that returns all the sub-arguments of argument (Ψ, ψ). For any $\mathcal{E} \subseteq \mathtt{Arg}(\Phi)$, $\mathtt{Concs}(\mathcal{E}) = \{\mathtt{Conc}(a) \mid a \in \mathcal{E}\}$ and $\mathtt{Base}(\mathcal{E}) = \bigcup_{a \in \mathcal{E}} \mathtt{Supp}(a)$. $\mathtt{Max}(\Phi)$ is the set of all maximal (for set inclusion) consistent subsets of Φ, i.e. for any $S \in \mathtt{Max}(\Phi)$, $S \subseteq \Phi$, S is consistent wrt logic $(\mathcal{F}, w, \mathtt{CN})$, and for any $\phi \in \Phi \setminus S$, it holds that $S \cup \{\phi\}$ is inconsistent. $\mathtt{MIC}(\Phi)$ denotes the set of all minimal (for set inclusion) inconsistent subsets of Φ, i.e., for any $S \in \mathtt{MIC}(\Phi)$, $S \subseteq \Phi$, S is inconsistent wrt logic $(\mathcal{F}, w, \mathtt{CN})$, and for any $\phi \in S$, it holds that $S \setminus \{\phi\}$ is consistent wrt logic $(\mathcal{F}, w, \mathtt{CN})$. Finally, $\mathtt{Free}(\Phi) = \{\phi \in \Phi \mid \forall S \in \mathtt{MIC}(\Phi), \phi \notin S\}$, i.e., $\mathtt{Free}(\Phi)$ is the set of formulae of Φ that are not involved in any minimal (for set inclusion) inconsistent subset of Φ.

Since information may be inconsistent, arguments may attack each other. In what follows, such attacks are captured by a binary relation, denoted by $\mathcal{R}$. For two arguments a, b, $(a, b) \in \mathcal{R}$ (or $a\mathcal{R}b$) means that a *attacks* b. For the sake of generality, $\mathcal{R}$ is left *unspecified*. It can thus be instantiated in various ways (see [19] for examples of instantiations of $\mathcal{R}$). However, we assume that it is based on inconsistency.

Definition 6 (Conflict-dependency) *Let $\langle \mathcal{F}, w, \mathtt{CN} \rangle$ be a logic and $\Phi \subseteq_f \mathcal{F}$. An attack relation $\mathcal{R} \subseteq \mathtt{Arg}(\Phi) \times \mathtt{Arg}(\Phi)$ is* conflict-dependent *iff for all $a, b \in \mathtt{Arg}(\Phi)$, if $(a, b) \in \mathcal{R}$ then $\mathtt{Supp}(a) \cup \mathtt{Supp}(b)$ is inconsistent.*

All existing attack relations are conflict-dependent with a notable exception, *undercutting* [24], which prevents the application of defaults in case of logics built on top of rule-based languages like ASPIC [4].

As said before, an argumentation-based logic defines from each set of formula a directed graph whose nodes are arguments and arrows are attacks between them.

Definition 7 (Argumentation function) *An* argumentation function $\mathcal{G}$ *on a logic* $\langle \mathcal{F}$, w, $\mathtt{CN}\rangle$ *transforms any set* $\Phi \subseteq_f \mathcal{F}$ *into a finite directed graph*[3] $\langle \mathtt{Arg}(\Phi), \mathcal{R}\rangle$ *where* $\mathcal{R} \subseteq \mathtt{Arg}(\Phi) \times \mathtt{Arg}(\Phi)$ *is a conflict-dependent attack relation.*

We are now ready to introduce *argumentation-based logics* (AL). An AL is a logic (in the sense of Definition 1) which is defined upon a *base logic*. The latter is supposed to behave in a rational way when information is consistent but exhibits an irrational behaviour in presence of inconsistency. Propositional logic is an example of such logic. AL restricts thus the base logic's inference power. An AL proceeds as follows: For any set Φ of formulas in the base logic, it defines its corresponding argumentation graph. The conclusions to be drawn from Φ using the consequence operator of the AL are the formulae that are supported by good arguments according to a given semantics $\mathcal{S}$.

Definition 8 (AL) *An* argumentation-based logic *(AL) is a logic* $\mathcal{L} = \langle \mathcal{F}, w, \mathtt{CN}'\rangle$ *which is based on base logic* $\langle \mathcal{F}, w, \mathtt{CN}\rangle$*, argumentation function* $\mathcal{G}$ *on* $\langle \mathcal{F}, w, \mathtt{CN}\rangle$ *and semantics* $\mathcal{S}$*, where for any* $\Phi \subseteq_f \mathcal{F}$, $\mathtt{CN}'(\Phi) \subseteq \{\phi \in \mathcal{F} \mid \exists a \in \mathtt{Arg}(\Phi)$ *with* $\mathcal{G}(\Phi) = \langle \mathtt{Arg}(\Phi), \mathcal{R}\rangle$ *and* $\mathtt{Conc}(a) \equiv \phi$ *wrt logic* $\langle \mathcal{F}, w, \mathtt{CN}\rangle\}$*. If* Φ *is consistent wrt* $\langle \mathcal{F}, w, \mathtt{CN}\rangle$*, then* $\mathtt{CN}'(\Phi) = \mathtt{CN}(\Phi)$*.*

In the next two sections, we define more precisely the consequence operators $\mathtt{CN}'$ of the logics induced by extensions semantics and ranking ones.

3 Logics induced by extension semantics

The most popular semantics were proposed by Dung in his seminal paper [15]. Those semantics as well as their refinements (e.g. in [5, 14]) partition the powerset of the set of arguments into two classes: *extensions* and *non-extensions*. Every extension represents a coherent point of view. We illustrate the kind of paraconsistent logics induced by such semantics, namely naive, stable and preferred. Before giving the formal definitions of the three semantics, we first introduce two key concepts on which they are based.

Definition 9 (Conflict-freeness–Defence) *Let* $\mathcal{T} = \langle \mathcal{A}, \mathcal{R}\rangle$ *be an argumentation graph,* $\mathcal{E} \subseteq \mathcal{A}$ *and* $a \in \mathcal{A}$*.*

- $\mathcal{E}$ *is* conflict-free *iff* $\nexists a, b \in \mathcal{E}$ *such that* $a\mathcal{R}b$*.*
- $\mathcal{E}$ defends *an argument* a *iff* $\forall b \in \mathcal{A}$ *such that* $b\mathcal{R}a$, $\exists c \in \mathcal{E}$ *such that* $c\mathcal{R}b$*.*

Definition 10 (Semantics) *Let* $\mathcal{T} = \langle \mathcal{A}, \mathcal{R}\rangle$ *be an argumentation graph and* $\mathcal{E} \subseteq \mathcal{A}$*.*

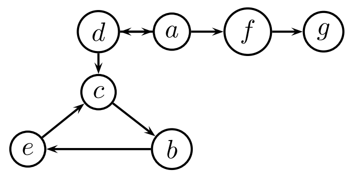
- $\mathcal{E}$ *is a* naive *extension iff it is a maximal (w.r.t. set* $\subseteq$*) conflict-free set.*
- $\mathcal{E}$ *is an* admissible *set iff it defends all its elements.*
- $\mathcal{E}$ *is a* complete *extension iff it is an admissible set that contains any argument it defends.*
- $\mathcal{E}$ *is a* preferred *extension iff it is a maximal (w.r.t. set* $\subseteq$*) set that is conflict-free and defends its elements.*
- $\mathcal{E}$ *is a* stable *extension iff it is conflict-free and attacks any argument in* $\mathcal{A} \setminus \mathcal{E}$*.*
- $\mathcal{E}$ *is a* grounded extension *iff it is a minimal (w.r.t. set* $\subseteq$*) complete extension.*

[3] In the literature, the pair $\langle \mathtt{Arg}(\Phi), \mathcal{R}\rangle$ is also called *argumentation system*.

– $\mathcal{E}$ *is an* ideal extension *iff it is a maximal (w.r.t. set* $\subseteq$*) admissible set contained in every preferred extension.*

Notations: $\mathtt{Ext}_x(\mathcal{T})$ denotes the set of all extensions of $\mathcal{T}$ under semantics x where $x \in \{n, p, s\}$ and n (resp. p and s) stands for naive (respectively preferred and stable). When we do not need to refer to a particular semantics, we write $\mathtt{Ext}(\mathcal{T})$ for short. Since any argumentation framework $\mathcal{T}$ has a single grounded and a single ideal extension, they will be denoted respectively by $\mathtt{GE}(\mathcal{T})$ and $\mathtt{IE}(\mathcal{T})$.

Example 1 *The argumentation graph depicted below*



has five naive extensions: $\mathcal{E}_1 = \{a, c, g\}$*,* $\mathcal{E}_2 = \{d, e, f\}$*,* $\mathcal{E}_3 = \{b, d, f\}$*,* $\mathcal{E}_4 = \{a, e, g\}$*,* $\mathcal{E}_5 = \{a, b, g\}$*; one stable* $\mathcal{E}_3$ *and two preferred extensions* $\mathcal{E}_3$ *and* $\mathcal{E}_6 = \{a, g\}$*.*

It is worth recalling that stable extensions are naive (respectively preferred) extensions but the converses are not always true. Moreover, an argumentation framework may have no stable extensions.

Let us now define the plausible conclusions that may be drawn from a set of formulae Φ by an argumentation-based logic. The idea is to infer a formula ϕ from Φ iff it is the conclusion of at least one argument in every extension of the argumentation graph built from Φ.

Definition 11 (ALES) *An* argumentation-based logic induced from extension semantics *(ALES) is a logic* $\mathcal{L} = \langle \mathcal{F}, w, \mathtt{CN}' \rangle$ *based on base logic* $\langle \mathcal{F}, w, \mathtt{CN} \rangle$*, argumentation function* $\mathcal{G}$*, and semantics* $x \in \{n, p, s\}$*, where*

$$\text{for all } \Phi \subseteq_f \mathcal{F}, \text{ for all } \phi \in \mathcal{F},$$
$$\phi \in \mathtt{CN}'(\Phi) \text{ iff } \forall \mathcal{E} \in \mathtt{Ext_x}(\mathcal{G}(\Phi)), \exists a \in \mathcal{E} \text{ s.t. } \mathtt{Conc}(a) \equiv x \text{ wrt logic } \langle \mathcal{F}, w, \mathtt{CN} \rangle.$$

In [3] a comprehensive study has been made on the family of logics described in this section. It has been shown that when the argumentation graph built over a set of formulae satisfies two key properties, then there is a full correspondence between the naive extensions of the graph and the maximal consistent subsets of the set of formulae. Before presenting the formal result, let us first recall the two properties.

Postulates (Closure under sub-arguments – Consistency) Let $\mathcal{G}$ be an argumentation function on logic $\langle \mathcal{F}, w, \mathtt{CN} \rangle$ and $\Phi \subseteq_f \mathcal{F}$. For all $\mathcal{E} \in \mathtt{Ext}(\mathcal{G}(\Phi))$,

– if $a \in \mathcal{E}$, then $\mathtt{Sub}(a) \subseteq \mathcal{E}$. We say that $\mathcal{G}(\Phi)$ is closed under sub-arguments.
– $\mathtt{Concs}(\mathcal{E})$ is consistent. We say that $\mathcal{G}(\Phi)$ satisfies consistency.

The following result shows that there is a one-to-one correspondence between the naive extensions of an argumentation graph and the maximal (for set inclusion) subsets of the set of formulae over which the graph is built. Indeed, each maximal consistent subset gives birth to a naive extension using the function $\mathtt{Arg}$ and each naive extension returns a maximal consistent set of formulae using the function $\mathtt{Base}$.

Theorem 1. *[3] Let $\mathcal{G}$ be an argumentation function on an adjunctive logic $\langle \mathcal{F}, w, \mathtt{CN} \rangle$, and let $\Phi \subseteq_f \mathcal{F}$. If $\mathcal{G}(\Phi)$ satisfies consistency and is closed under sub-arguments (under naive semantics), then:*

- *For all $\mathcal{E} \in \mathtt{Ext}_n(\mathcal{G}(\Phi))$, $\mathtt{Base}(\mathcal{E}) \in \mathtt{Max}(\Phi)$.*
- *For all $\mathcal{E}_i, \mathcal{E}_j \in \mathtt{Ext}_n(\mathcal{G}(\Phi))$, if $\mathtt{Base}(\mathcal{E}_i) = \mathtt{Base}(\mathcal{E}_j)$ then $\mathcal{E}_i = \mathcal{E}_j$.*
- *For all $\mathcal{E} \in \mathtt{Ext}_n(\mathcal{G}(\Phi))$, $\mathcal{E} = \mathtt{Arg}(\mathtt{Base}(\mathcal{E}))$.*
- *For all $\mathcal{S} \in \mathtt{Max}(\Phi)$, $\mathtt{Arg}(\mathcal{S}) \in \mathtt{Ext}_n(\mathcal{G}(\Phi))$.*

Let us now characterize the set of inferences that may be drawn from a set of formulae Φ by any argumentation-based logic under naive semantics. It coincides with the set of inferences that are drawn from the maximal consistent subsets of Φ.

Theorem 2. *[3] Let $\mathcal{L} = \langle \mathcal{F}, w, \mathtt{CN}' \rangle$ be an ALES based on adjunctive logic $\langle \mathcal{F}, w, \mathtt{CN} \rangle$, argumentation function $\mathcal{G}$, and naive semantics. For all $\Phi \subseteq_f \mathcal{F}$, if $\mathcal{G}(\Phi)$ satisfies consistency and is closed under sub-arguments, then*

$$\mathtt{CN}'(\Phi) = \bigcap_{\mathcal{S}_i \in \mathtt{Max}(\Phi)} \mathtt{CN}(\mathcal{S}_i).$$

It is worth noticing that the paraconsistent logics ALES restore consistency and return flat consequences. Moreover, they generalize to any Tarskian logic the universal logic defined by Rescher and Manor in [27].

In both previous theorems, the base logic is considered adjunctive. An important question is what about the case where the base logic is not adjunctive? It was shown in [3] that in that case the argumentation-based logic may choose arbitrarily *some* maximal consistent subsets of a knowledge base, leading thus to counter-intuitive outcomes.

A similar study has been conducted for stable and preferred semantics. It has been shown that there are two families of attack relations. The first family leads to coherent argumentation graphs (i.e., their stable extensions coincide with their preferred ones). Furthermore, stable extensions coincide with the naive ones. Such graphs coincide then with the above discussed ones. The ideal extension of these graphs coincide with the grounded extension. It also coincides with the intersection of the preferred (thus naive, stable) extensions. Furthermore, it is exactly the set $\mathtt{Arg}(\mathtt{Free}(\Phi))$, i.e., the set of arguments built over the set of free formulae of Φ. Finally, the conclusions drawn under ideal and grounded semantics are the formulae that follow using the base logic from the free formulae of a given set of formulae.

The second family of attack relations allows choosing only *some* maximal consistent subbsets of a knowledge base. Thus, the corresponding argumentation-based logics infer counter-intuitive conclusions under stable and preferred semantics. The grounded and ideal extensions of such graphs may lead to counter-intuitive results as well.

Let us now illustrate this approach with propositional logic, an instance of Tarskian logic. We assume that there is a *finite number of variables* in the language. This assumption, very common in the literature, ensures the finiteness condition of Definition 1. The attack relation between arguments is *assumption-attack* introduced for the first time in [16].

Definition 12 (Assumption attack) *Let $\langle \mathcal{F}, w, \mathtt{CN} \rangle$ be propositional logic. An argument $\langle \Psi, \psi \rangle$* attacks *an argument $\langle \Psi', \psi' \rangle$, denoted by $\langle \Psi, \psi \rangle \ \mathcal{R}_{as} \ \langle \Psi', \psi' \rangle$, iff $\exists \phi \in \Psi'$ s.t. $\psi \equiv \neg \phi$.*

Assumption attack is among the attacks relation that lead to rational argumentation-based logics.

Theorem 3. *[3] Let $\mathcal{G}$ be an argumentation function on propositional logic $\mathcal{L} = \langle \mathcal{F}, w, \mathtt{CN} \rangle$ such that for all $\Phi \subseteq_f \mathcal{F}$, $\mathcal{G}(\Phi) = \langle \mathtt{Arg}(\Phi), \mathcal{R}_{as} \rangle$. It holds that*

$$\mathtt{Ext}_n(\mathcal{G}(\Phi)) = \mathtt{Ext}_s(\mathcal{G}(\Phi)) = \mathtt{Ext}_p(\mathcal{G}(\Phi)).$$

$$\mathtt{IE}(\mathcal{G}(\Phi)) = \mathtt{GE}(\mathcal{G}(\Phi)) = \mathtt{Arg}(\mathtt{Free}(\Phi)).$$

Let us now consider the following example.

Example 2 *Let $\Phi = \{p, \neg p, q, p \to \neg q\}$ be a propositional knowledge base. This base has three maximal (for set inclusion) consistent subbases:*

- $\Phi_1 = \{p, q\}$,
- $\Phi_2 = \{p, p \to \neg q\}$,
- $\Phi_3 = \{\neg p, q, p \to \neg q\}$.

An argumentation-based logic induced from naive, stable, or preferred semantics will draw from Φ the tautologies since they are the only common consequences of the three subbases.

Note that none of the two conflicts $\{p, \neg p\}$ and $\{p, q, p \to \neg q\}$ is solved. Such output may seem unsatisfactory in general and in multi-agent systems where one needs an efficient way for solving conflicts between agents. Let us now have a closer look at the knowledge base Φ. The four formulae in Φ do not have the same responsibility for inconsistency. For instance, the degree of blame of p is higher than the one of q since it is involved in more conflicts. Moreover, p is frontally opposed while q is opposed in an indirect way. Similarly, $\neg q$ is more to blame than q since it follows from the controversial formula p.

To sum up, *rational* argumentation-based logics induced from extension semantics generalize, to any Tarskian logic, the paraconsistent logic defined by Rescher and Manor in [27] on top of propositional logic. Such logics coincide with their base logic in case of a consistent knowledge base. When the latter is inconsistent, they only draw the formulae that follow logically (using the base logic) from the set of formulae which are not involved in inconsistency. This means that they leave conflicts unsolved as shown in Example 2.

4 Logics induced by ranking semantics

Ranking semantics have been introduced in [1] as an alternative approach for evaluating arguments. Their basic idea is to rank arguments from the most to the less acceptable ones, instead of computing extensions. They should satisfy axioms (i.e., desirable properties), some of them are mandatory while others are optional. In what follows, we investigate the argumentation-based logics induced from *burden-based semantics* (Bbs), a ranking semantics introduced in [1]. Bbs assigns a *burden number* to every argument. The heavier the burden of an argument, the weaker its attacks.

Definition 13 (Burden numbers) *Let* $\mathcal{T} = \langle \mathcal{A}, \mathcal{R} \rangle$ *be a finite argumentation graph,* $i \in \{0, 1, \ldots\}$*, and* $a \in \mathcal{A}$*. We denote by* $\mathtt{Bur}_i(a)$ *the* burden number *of* a *in the* i^{th} *step:*

$$\mathtt{Bur}_i(a) = \begin{cases} 1 & \text{if } i = 0; \\ 1 + \sum_{b \in \mathtt{Att}(a)} \frac{1}{\mathtt{Bur}_{i-1}(b)} & \text{otherwise}. \end{cases}$$

where $\mathtt{Att}(a) = \{b \in \mathcal{A} \mid (b, a) \in \mathcal{R}\}$.

By convention, if $\mathtt{Att}(a) = \emptyset$, then

$$\sum_{b \in \mathtt{Att}(a)} \frac{1}{\mathtt{Bur}_{i-1}(b)} = 0.$$

Let us illustrate this function in the following example.

Example 1 (Cont) Consider the argumentation graph depicted in Example 1. The burden numbers of each argument are summarized in the table below.

Step i	a	b	c	d	e	f	g
0	1	1	1	1	1	1	1
1	2	2	3	2	2	2	2
2	1.5	1.33	2	1.5	1.5	1.5	1.5
3	1.66	1.5	2.33	1.66	1.75	1.66	1.66
4	1.60	1.42	2.17	1.60	1.66	1.60	1.60
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

It is worth pointing out that the function $\mathtt{Bur}$ converges, and thus each argument a has a single burden number $\mathtt{Bur}(a) = \lim_{i \to \infty} \mathtt{Bur}_i(a)$.

There are different ways of comparing pairs of arguments, each of which leads to a new semantics. For instance, one may compare the final burden numbers of arguments (i.e., the ones got by the limit). The corresponding semantics satisfies all the mandatory axioms defined in [1]. Moreover, it allows *compensation*. Indeed, it considers that two weak attacks are equivalent to a strong one. Another alternative consists of comparing arguments lexicographically as follows:

Definition 14 (Bbs) *The* burden-based semantics $\mathtt{Bbs}$ *transforms any argumentation graph* $\mathcal{T} = \langle \mathcal{A}, \mathcal{R} \rangle$ *into the ranking* $\mathtt{Bbs}(\mathcal{T})$ *on* $\mathcal{A}$ *such that* $\forall a, b \in \mathcal{A}$, $\langle a, b \rangle \in \mathtt{Bbs}(\mathcal{T})$ *iff one of the two following cases holds:*

- $\forall\, i \in \{0, 1, \ldots\}$, $\texttt{Bur}_i(a) = \texttt{Bur}_i(b)$;
- $\exists\, i \in \{0, 1, \ldots\}$, $\texttt{Bur}_i(a) < \texttt{Bur}_i(b)$ *and* $\forall\, j \in \{0, 1, \ldots, i-1\}$, $\texttt{Bur}_j(a) = \texttt{Bur}_j(b)$.

Intuitively, $\langle a, b\rangle \in \texttt{Bbs}(\mathcal{T})$ means that a is *at least as acceptable* as b. Let us see in an example how the semantics works.

Example 1 (Cont) According to $\texttt{Bbs}$, the argument b is strictly more acceptable than a, d, f, and g which are themselves equally acceptable and strictly more acceptable than e. Finally, e is more acceptable than c.

It is worth noticing that Bbs considers finite argumentation graphs. However, from a finite set of formulae, Definition 5 may generate an infinite number of arguments. Of course, most of them are redundant. In order to avoid such useless arguments, we assume that a logic satisfies the following additional condition:

– $\{\texttt{CN}(\{\phi\}) \mid \phi \in \mathcal{F}\}$ is finite (Finiteness)

Finiteness ensures a finite number of non-equivalent formulae. This condition, not considered by Tarski, will avoid redundant arguments. It is worth recalling that classical logic satisfies finiteness when the number of propositional variables is finite, which is a quite common assumption in the literature.

Property 1. For all $\Phi \subseteq_f \mathcal{F}$, $\texttt{Arg}(\Phi)$ is finite.

The plausible conclusions of an argumentation-based logic that uses ranking semantics are simply those supported by at least one argument. Note that a formula and its negation may both be plausible. This means that the approach tolerates inconsistency. More importantly, the conclusions are ranked from the most to the least plausible ones. A formula is ranked higher than another formula if it is supported by an argument which is more acceptable than any argument supporting the second formula. The notation $\phi \succeq \psi$ means that ϕ is at least as plausible as ψ.

Definition 15 (ALRS) *An* argumentation-based logic induced from ranking semantics *(ALRS) is a logic $\mathcal{L} = \langle \mathcal{F}, w, \texttt{CN}'\rangle$ based on base logic $\langle \mathcal{F}, w, \texttt{CN}\rangle$, argumentation function $\mathcal{G}$, and semantics* $\texttt{Bbs}$, *where for all $\Phi \subseteq_f \mathcal{F}$, $\mathcal{G}(\Phi) = \langle \texttt{Arg}(\Phi), \mathcal{R}\rangle$, and*

– $\texttt{CN}'(\Phi) = \{\phi \in \mathcal{F} \mid \exists a \in \texttt{Arg}(\Phi)$ *and* $\texttt{Conc}(a) \equiv \phi$ *wrt logic* $\langle \mathcal{F}, w, \texttt{CN}\rangle\}$.
– *for all* $\phi, \psi \in \texttt{CN}'(\Phi)$, $\phi \succeq \psi$ *iff* $\exists a \in \texttt{Arg}(\Phi)$ *such that* $\texttt{Conc}(a) \equiv \phi$ *and* $\forall b \in \texttt{Arg}(\Phi)$ *such that* $\texttt{Conc}(b) \equiv \psi$, $\langle a, b\rangle \in \texttt{Bbs}(\mathcal{G}(\Phi))$.

Unlike certain well-known inconsistency-tolerating logics (like the 3- and 4-valued ones [6, 13]), the above logics satisfy the following crucial property: if the premises are consistent, the conclusions coincide with those of CN. They satisfy other important properties like ranking free formulae above non-free ones. Recall that free formulae are those that are not involved in any minimal (for set inclusion) inconsistent subset of a knowledge base. They also consider that any formula is at most as plausible as its logical consequences and thus equivalent formula are equally plausible (see [2] for a complete study of these logics).

Let us now illustrate this family of logics by considering propositional logic as base logic and assumption-attack as attack relation.

Example 2 (Cont) Let $\Phi = \{p, \neg p, q, p \rightarrow \neg q\}$ be a propositional knowledge base. Assume that the non-equivalent formulae selected by the well-ordering w are as follows:

$p \wedge \neg p$	$\neg p \wedge \neg q$	$\neg p$	$p \rightarrow \neg q$	$p \vee \neg p$
	$\neg p \wedge q$	$\neg q$	$p \rightarrow q$	
	$p \wedge \neg q$	$p \leftrightarrow q$	$q \rightarrow p$	
	$p \wedge q$	$p \leftrightarrow \neg q$	$\neg p \rightarrow q$	
		q		
		p		

The set $\mathtt{Arg}(\Phi)$ contains the 21 following arguments.

a	$\langle\{\neg p, q\}, \neg p \wedge q\rangle$	k	$\langle\{q, p \rightarrow \neg q\}, p \leftrightarrow \neg q\rangle$
b	$\langle\{q, p \rightarrow \neg q\}, \neg p \wedge q\rangle$	l	$\langle\{q\}, q\rangle$
c	$\langle\{p, p \rightarrow \neg q\}, p \wedge \neg q\rangle$	m	$\langle\{p\}, p\rangle$
d	$\langle\{p, q\}, p \wedge q\rangle$	n	$\langle\{\neg p\}, p \rightarrow \neg q\rangle$
e	$\langle\{\neg p\}, \neg p\rangle$	o	$\langle\{p \rightarrow \neg q\}, p \rightarrow \neg q\rangle$
f	$\langle\{q, p \rightarrow \neg q\}, \neg p\rangle$	y	$\langle\{\neg p\}, p \rightarrow q\rangle$
g	$\langle\{p, p \rightarrow \neg q\}, \neg q\rangle$	z	$\langle\{q\}, p \rightarrow q\rangle$
h	$\langle\{p, q\}, p \leftrightarrow q\rangle$	r	$\langle\{p\}, q \rightarrow p\rangle$
i	$\langle\{\neg p, q\}, p \leftrightarrow \neg q\rangle$	s	$\langle\{p\}, \neg p \rightarrow q\rangle$
j	$\langle\{p, p \rightarrow \neg q\}, p \leftrightarrow \neg q\rangle$	t	$\langle\{q\}, \neg p \rightarrow q\rangle$
		u	$\langle\emptyset, p \vee \neg p\rangle$

The set $\mathtt{CN}'(\Phi)$ contains the conclusions of the arguments and their equivalent formulae. Due to the large number of attacks, we do not give them here. From the argumentation graph, the following burden numbers are computed (table on the left). The ranking on $\mathtt{Arg}(\Phi)$ is as shown in the table on the right.

	$i=0$	$i=1$	$i=2$	$i=3$
u	1	1	1	1
m, s, r	1	3	1.83	2.41
e, n, y	1	2	1.33	1.54
l, z, t	1	2	1.25	1.48
o	1	2	1.25	1.48
d, h	1	4	2.05	2.89
b, f, k	1	3	1.50	1.96
c, g, j	1	4	2.05	2.89
a, i	1	3	1.58	2.02

u
o, l, z, t
e, n, y
b, f, k
a, i
m, r, s
c, d, g, h, j

The conclusions of the arguments are ranked as follows:

$p \vee \neg p$
$p \rightarrow \neg q,\ q,\ p \rightarrow q,\ \neg p \rightarrow q$
$\neg p$
$\neg p \wedge q,\ p \leftrightarrow \neg q$
$p,\ q \rightarrow p$
$p \wedge \neg q,\ \neg q,\ p \wedge q,\ p \leftrightarrow q$

Recall that equivalent formula are equally plausible. For instance, $p \wedge p$ is as plausible as p. Note that $\neg p$ is more plausible than p, and q is more plausible than $\neg q$. Thus, unlike ALES, logics that use ranking semantics solve both conflicts of the base Φ.

The ranking of formulae produced by the previous logic is not arbitrary. It not only satisfies some rationality postulates discussed in [2], it also captures in some cases a well-known *inconsistency measure* [20]. The latter assigns a degree of blame to each formula of a knowledge base. This degree is the number of minimal inconsistent susbsets of the base (called conflicts) in which the formula is involved. It was shown in [2] that if a formula ϕ of a knowledge base is involved in more conflicts than another formula ψ of the base, then ψ is more plausible than ϕ. This result is only true in case each formula in the base cannot be inferred from another consistent subset of the base.

Theorem 4. *[2] Let $\mathcal{L} = \langle \mathcal{F}, w, \mathtt{CN}' \rangle$ be an ALRS based on propositional logic $\langle \mathcal{F}, w, \mathtt{CN} \rangle$, argumentation function $\mathcal{G}$ such that for all $\Phi \subseteq_f \mathcal{F}$, $\mathcal{G}(\Phi) = \langle \mathtt{Arg}(\Phi), \mathcal{R}_{as} \rangle$, and semantics* Bbs. *Let $\Phi \subseteq \mathcal{F}$. If for all $\phi \in \Phi$, $\nexists \Psi \subseteq \Phi \setminus \{\phi\}$ such that Ψ is consistent and $\phi \in \mathtt{CN}(\Psi)$, then for all $\phi, \psi \in \mathtt{CN}'(\Phi) \cap \Phi$, if $|\{\Psi \in \mathtt{MIC}(\Phi) \mid \phi \in \Psi\}| > |\{\Psi' \in \mathtt{MIC}(\Phi) \mid \psi \in \Psi'\}|$, then $\psi \succ \phi$ (i.e., $\psi \succeq \phi$ and $\phi \not\succeq \psi$).*

Works on inconsistency measures focus only on the formulae of the base and completely neglect their logical consequences. ALRS focus on both. That's why the two approaches may not find the same results in the general case. Indeed, it may be the case that a formula ϕ of a base is involved in more conflicts than another formula ψ of the same base, but ϕ follows logically from a subset of the base and this subset constitutes a more acceptable argument than the one supporting ψ. Thus, ALRS will rank ϕ higher than ψ while the inconsistency measure will prefer ψ.

To sum up, argumentation-based logics induced from ranking semantics tolerate inconsistency in that they may infer inconsistent conclusions. Furthermore, they rank-order the conclusions with regard to plausibility. Finally, unlike ALES, they solve inconsistency.

5 Conclusion

Argumentation is a natural approach for handling inconsistency. It is more akin to the way humans deal with inconsistency in everyday life. Indeed, it constructs arguments pro and arguments con claims, then it evaluates the arguments before concluding.

This paper discussed two families of argumentation-based paraconsistent logics: the family of logics that use naive (respectively stable, preferred, grounded and ideal) semantics, and the family of logics that use ranking semantics, namely Bbs. We have shown that argumentation logics are efficient since, under extension semantics, they generalize well-known logics and, under ranking semantics, they outperform them.

References

1. L. Amgoud and J. Ben-Naim. Ranking-based semantics for argumentation frameworks. In *SUM*, pages 134–147, 2013.

2. L. Amgoud and J. Ben-Naim. Argumentation-based ranking logics. In *Proceedings of the 2015 International Conference on Autonomous Agents and Multiagent Systems, (AAMAS'2015)*, pages 1511–1519, 2015.
3. L. Amgoud and P. Besnard. Logical limits of abstract argumentation frameworks. *Journal of Applied Non-Classical Logics*, 23(3):229–267, 2013.
4. L. Amgoud, M. Caminada, C. Cayrol, M.-C. Lagasquie, and H. Prakken. Towards a consensual formal model: inference part. *Deliverable of ASPIC project*, 2004.
5. P. Baroni, M. Giacomin, and G. Guida. Scc-recursiveness: a general schema for argumentation semantics. *Artificial Intelligence Journal*, 168:162–210, 2005.
6. N. D. Belnap. A Useful Four-Valued Logic. In J. Dunn and G. Epstein, editors, *Modern Uses of Multiple-Valued Logic*, pages 7–37. Oriel Press, 1977.
7. S. Benferhat, D. Dubois, and H. Prade. How to infer from inconsistent beliefs without revising? In *14th International Joint Conference on Artificial Intelligence (IJCAI'95)*, pages 1449–1455, 1995.
8. P. Besnard and A. Hunter. A logic-based theory of deductive arguments. *Artificial Intelligence*, 128(1-2):203–235, 2001.
9. D. G. Bobrow. Special issue on non-monotonic reasoning. *Artificial Intelligence Journal*, 13 (1-2), 1980.
10. G. Brewka. Preferred subtheories: an extended logical framework for default reasoning. In *Proceedings of the 11th International Joint Conference on Artificial Intelligence, (IJCAI'89)*, pages 1043–1048, 1989.
11. C. Cayrol and M.-C. Lagasquie-Schiex. Graduality in Argumentation. *Journal of Artificial Intelligence Research (JAIR)*, 23:245–297, 2005.
12. L. Cholvy. Automated reasoning with merged contradictory information whose reliability depends on topics. In *European Conference on Symbolic and Quantitative Approaches to Reasoning under Uncertainty, ECSQARU'95*, pages 125–132, 1995.
13. I. D'Ottaviano and N. da Costa. Sur un problème de Jaśkowski. In *Comptes Rendus de l'Académie des Sciences de Paris*, volume 270, pages 1349–1353, 1970.
14. P. Dung, P. Mancarella, and F. Toni. Computing ideal skeptical argumentation. *Artificial Intelligence Journal*, 171:642–674, 2007.
15. P. M. Dung. On the Acceptability of Arguments and its Fundamental Role in Non-Monotonic Reasoning, Logic Programming and n-Person Games. *AIJ*, 77:321–357, 1995.
16. M. Elvang-Gøransson, J. Fox, and P. Krause. Acceptability of arguments as 'logical uncertainty'. In *2nd European Conference on Symbolic and Quantitative Approaches to Reasoning and Uncertainty (ECSQARU'93)*, volume 747 of *Lecture Notes in Computer Science*, pages 85–90, Granada, Spain, November 1993. Springer.
17. M. Fitting. Kleene's three-valued logics and their children. *Fundamenta Informaticae*, 20:113–131, 1994.
18. D. Gabbay and A. Hunter. Making inconsistency respectable: a logical framework for inconsistency in reasoning. *Fundamentals of Artificial Intelligence Research*, LNAI 535:19–32, 1991.
19. N. Gorogiannis and A. Hunter. Instantiating abstract argumentation with classical logic arguments: Postulates and properties. *Artificial Intelligence*, 175(9-10):1479–1497, 2011.
20. A. Hunter and S. Konieczny. On the measure of conflicts: Shapley inconsistency values. *Artificial Intelligence*, 174(14):1007–1026, 2010.
21. J. D. Kleer. Using crude probability estimates to guide diagnosis. *Artificial Intelligence*, 45:381–391, 1990.
22. P. Matt and F. Toni. A game-theoretic measure of argument strength for abstract argumentation. In *Proceedings of 11th European Conference on Logics in Artificial Intelligence, JELIA'08*, volume 5293 of *Lecture Notes in Computer Science*, pages 285–297. Springer, 2008.

23. J. Pearl. Qualitative probabilities for default reasoning, belief revision, and causal modeling. *Artifitial Intelligence*, 84(1-2):57–112, 1996.
24. J. Pollock. How to reason defeasibly. *Artificial Intelligence J.*, 57(1):1–42, 1992.
25. F. Pu, J. Luo, Y. Zhang, and G. Luo. Argument ranking with categoriser function. In *Knowledge Science, Engineering and Management - 7th International Conference, KSEM 2014, Proceedings*, pages 290–301, 2014.
26. R. Reiter. A logic for default reasoning. *Artificial Intelligence*, 13(1-2):81–132, 1980.
27. N. Rescher and R. Manor. On inference from inconsistent premises. *Journal of Theory and decision*, 1:179–219, 1970.
28. A. Tarski. *Logic, Semantics, Metamathematics (E. H. Woodger, editor))*, chapter On Some Fundamental Concepts of Metamathematics. Oxford Uni. Press, 1956.
29. M. Thimm and G. Kern-Isberner. On controversiality of arguments and stratified labelings. In *Proceedings of Computational Models of Argument, (COMMA'2014)*, pages 413–420, 2014.