Stable semantics in logic-based argumentation

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Abstract. This paper investigates the outputs of abstract logic-based argumentation systems under *stable* semantics. We delimit the number of stable extensions a system may have. We show that in the best case, an argumentation system infers exactly the common conclusions drawn from the maximal consistent subbases of the original knowledge base. This output corresponds to that returned by a system under the naive semantics. In the worst case, counter-intuitive results are returned. In the intermediary case, the system forgets intuitive conclusions. These two latter cases are due to the use of skewed attack relations. The results show that stable semantics is either useless or unsuitable in logic-based argumentation systems. Finally, we show that under this semantics, argumentation systems may inherit the problems of coherence-based approaches.

1 Introduction

An argumentation system for reasoning with inconsistent knowledge is built from a knowledge base using a monotonic logic. It consists of a set of *arguments*, *attacks* among them and a *semantics* for evaluating the arguments (see [4, 10, 7, 12] for some examples of such systems).

Stable semantics is one of the prominent semantics proposed in [8]. A set of arguments is acceptable (or an *extension*) under this semantics, if it is free of conflicts and attacks any argument outside the set. Note that this semantics does not guarantee the existence of extensions for a system. In [6], the author studied the kind of outputs that may be returned under this semantics. However, the focus was on *one particular* argumentation system: it is grounded on propositional logic and uses 'assumption attack' [9]. The results show that each stable extension of the system is built from one maximal consistent subbase of the original knowledge base. However, it is not clear whether this is true for other attack relations or other logics. It is neither clear whether systems that have stable extensions return intuitive results. It is also unclear what is going wrong with systems that do not have stable extensions. Finally, the number of stable extensions that a system may have is unknown.

In this paper, we conduct an in-depth study on the outputs of argumentation systems under stable semantics. We consider *abstract* logic-based systems, i.e., systems that use *Tarskian logics* [15] and *any attack relation*. For the first time, the maximum number of stable extensions a system may have is delimited. It is the number of maximal (for set inclusion) consistent subbases of the knowledge base. Moreover, we show that stable semantics is either useless or unsuitable for these systems. Indeed, in the best case, such systems infer exactly the conclusions that are drawn from all the maximal consistent subbases. This corresponds exactly to the output of the same systems under naive semantics. In the worst case, counter-intuitive results are returned. There is a third case where intuitive conclusions may be forgotten by the systems. These two last cases are due to the use of skewed attack relations. Finally, we show that argumentation systems that use stable semantics inherit the problems of coherence-based approaches [14].

The paper is organized as follows: Section 2 defines the logic-based argumentation systems we are interested in. Section 3 recalls three basic postulates that such systems should obey. Section 4 investigates the outcomes that are computed under stable semantics. Section 5 compares our work with existing ones and Section 6 concludes.

2 Logic-based argumentation systems

Argumentation systems are built on an underlying *monotonic logic*. In this paper, we focus on Tarski's monotonic logics [15]. Indeed, we consider logics ($\mathcal{L}$, $\mathtt{CN}$) where $\mathcal{L}$ is a set of well-formed *formulas* and $\mathtt{CN}$ is a *consequence operator*. It is a function from $2^{\mathcal{L}}$ to $2^{\mathcal{L}}$ which returns the set of formulas that are logical consequences of another set of formulas according to the logic in question. It satisfies the following basic properties:

1. $X \subseteq \mathtt{CN}(X)$ (Expansion)
2. $\mathtt{CN}(\mathtt{CN}(X)) = \mathtt{CN}(X)$ (Idempotence)
3. $\mathtt{CN}(X) = \bigcup_{Y \subseteq_f X} \mathtt{CN}(Y)$[1] (Finiteness)
4. $\mathtt{CN}(\{x\}) = \mathcal{L}$ for some $x \in \mathcal{L}$ (Absurdity)
5. $\mathtt{CN}(\emptyset) \neq \mathcal{L}$ (Coherence)

A $\mathtt{CN}$ that satisfies the above properties is *monotonic*. The associated notion of *consistency* is defined as follows:

Definition 1 (Consistency) *A set $X \subseteq \mathcal{L}$ is* consistent *wrt a logic* $(\mathcal{L}, \mathtt{CN})$ *iff* $\mathtt{CN}(X) \neq \mathcal{L}$. *It is* inconsistent *otherwise.*

Arguments are built from a *knowledge base* $\Sigma \subseteq \mathcal{L}$ as follows:

Definition 2 (Argument) *Let Σ be a knowledge base. An* argument *is a pair* (X, x) *s.t. $X \subseteq \Sigma$, X is consistent, and $x \in \mathtt{CN}(X)$*[2]*. An argument (X, x) is a* sub-argument *of (X', x') iff $X \subseteq X'$.*

Notations: $\mathtt{Supp}$ and $\mathtt{Conc}$ denote respectively the *support* X and the *conclusion* x of an argument (X, x). For all $\mathcal{S} \subseteq \Sigma$, $\mathtt{Arg}(\mathcal{S})$ denotes the set of all arguments that can be built from $\mathcal{S}$ by means of Definition 2. $\mathtt{Sub}$ is a function that returns all the sub-arguments of a given argument. For all $\mathcal{E} \subseteq \mathtt{Arg}(\Sigma)$, $\mathtt{Concs}(\mathcal{E}) = \{\mathtt{Conc}(a) \mid a \in \mathcal{E}\}$

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[1] $Y \subseteq_f X$ means that Y is a finite subset of X.

[2] Generally, the support X is minimal (for set inclusion). In this paper, we do not need to make this assumption.

and $\mathtt{Base}(\mathcal{E}) = \bigcup_{a \in \mathcal{E}} \mathtt{Supp}(a)$. $\mathtt{Max}(\Sigma)$ is the set of all maximal (for set inclusion) consistent subbases of Σ. Finally, $\mathtt{Free}(\Sigma) = \bigcap \mathcal{S}_i$ where $\mathcal{S}_i \in \mathtt{Max}(\Sigma)$, and $\mathtt{Inc}(\Sigma) = \Sigma \setminus \mathtt{Free}(\Sigma)$.

An argumentation system is defined as follows.

Definition 3 (Argumentation system) *An* argumentation system *(AS) over a knowledge base Σ is a pair $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ such that $\mathcal{R} \subseteq \mathtt{Arg}(\Sigma) \times \mathtt{Arg}(\Sigma)$ is an* attack *relation. For $a, b \in \mathtt{Arg}(\Sigma)$, $(a, b) \in \mathcal{R}$ (or $a\mathcal{R}b$) means that a attacks b.*

The attack relation is left *unspecified* in order to keep the system very general. It is also worth mentioning that the set $\mathtt{Arg}(\Sigma)$ may be infinite even when the base Σ is finite. This would mean that the argumentation system may be *infinite*[3]. Finally, arguments are evaluated using stable semantics.

Definition 4 (Stable semantics [8]) *Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an AS over a knowledge base Σ, and $\mathcal{E} \subseteq \mathtt{Arg}(\Sigma)$ s.t. $\nexists\, a, b \in \mathcal{E}$ s.t. $a\mathcal{R}b$.*

- *$\mathcal{E}$ is a* naive *extension iff $\mathcal{E}$ is maximal (for set inclusion).*
- *$\mathcal{E}$ is a* stable *extension iff $\forall a \in \mathtt{Arg}(\Sigma) \setminus \mathcal{E}$, $\exists b \in \mathcal{E}$ s.t. $b\mathcal{R}a$.*

It is worth noticing that each stable extension is a naive one but the converse is false. Let $\mathtt{Ext}_x(\mathcal{T})$ denote the set of all extensions of $\mathcal{T}$ under semantics x (n and s will stand respectively for naive and stable semantics). When we do not need to specify the semantics, we use the notation $\mathtt{Ext}(\mathcal{T})$ for short.

The extensions are used in order to define the conclusions to be drawn from Σ according to an argumentation system $\mathcal{T}$. The idea is to infer a formula x from Σ iff x is the conclusion of an argument in each extension. $\mathtt{Output}(\mathcal{T})$ is the set of all such formulas.

Definition 5 (Output) *Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an AS over a knowledge base Σ.* $\mathtt{Output}(\mathcal{T}) = \{x \in \mathcal{L} \mid \forall \mathcal{E} \in \mathtt{Ext}(\mathcal{T}), \exists a \in \mathcal{E}$ *s.t.* $\mathtt{Conc}(a) = x\}$.

$\mathtt{Output}(\mathcal{T})$ coincides with the set of common conclusions of the extensions. Indeed, $\mathtt{Output}(\mathcal{T}) = \bigcap \mathtt{Concs}(\mathcal{E}_i)$, $\mathcal{E}_i \in \mathtt{Ext}(\mathcal{T})$. Note also that when the base Σ contains only inconsistent formulas, then $\mathtt{Arg}(\Sigma) = \emptyset$. Consequently, $\mathtt{Ext}_s(\mathcal{T}) = \{\emptyset\}$ and $\mathtt{Output}(\mathcal{T}) = \emptyset$. Without loss of generality, throughout the paper, we assume that Σ is *finite* and contains at least one consistent formula.

3 Postulates for argumentation systems

In [5], it was argued that logic-based argumentation systems should obey to some rationality postulates, i.e., desirable properties that any reasoning system should enjoy. The three postulates proposed in [5] are revisited and extended to any Tarskian logic in [1]. The first one concerns the closure of the system's output under the consequence operator $\mathtt{CN}$. The idea is that the formalism should not forget conclusions.

[3] An AS is *finite* iff each argument is attacked by a finite number of arguments. It is *infinite* otherwise.

Postulate 1 (Closure under $\mathtt{CN}$**)** *Let* $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ *be an AS over a knowledge base* Σ. $\mathcal{T}$ *satisfies* closure *iff for all* $\mathcal{E} \in \mathtt{Ext}(\mathcal{T})$, $\mathtt{Concs}(\mathcal{E}) = \mathtt{CN}(\mathtt{Concs}(\mathcal{E}))$.

The second rationality postulate ensures that the acceptance of an argument should imply also the acceptance of all its sub-arguments.

Postulate 2 (Closure under sub-arguments) *Let* $\mathcal{T}$ = $(\mathtt{Args}(\Sigma)$, $\mathcal{R})$ *be an AS over a knowledge base* Σ. $\mathcal{T}$ *is* closed under sub-arguments *iff for all* $\mathcal{E} \in \mathtt{Ext}(\mathcal{T})$*, if* $a \in \mathcal{E}$*, then* $\mathtt{Sub}(a) \subseteq \mathcal{E}$.

The third rationality postulate ensures that the set of conclusions supported by each extension is consistent.

Postulate 3 (Consistency) *Let* $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ *be an AS over a knowledge base* Σ. $\mathcal{T}$ *satisfies* consistency *iff for all* $\mathcal{E} \in \mathtt{Ext}(\mathcal{T})$*,* $\mathtt{Concs}(\mathcal{E})$ *is consistent.*

In [1], the conditions under which these postulates are satisfies/violated are investigated. It is shown that the attack relation should be grounded on inconsistency. This is an obvious requirement especially for reasoning about inconsistent information.

Definition 6 (Conflict-dependent) *Let* $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ *be an AS. The attack relation* $\mathcal{R}$ *is* conflict-dependent *iff* $\forall a, b \in \mathtt{Arg}(\Sigma)$*, if* $a\mathcal{R}b$ *then* $\mathtt{Supp}(a) \cup \mathtt{Supp}(b)$ *is inconsistent.*

4 The outcomes of argumentation systems

As seen before, the acceptability of arguments is defined without considering neither their internal structure nor their origin. In this section, we fully characterize for the first time the 'concrete' outputs of an argumentation system under stable semantics. For that purpose, we consider only systems that enjoy the three rationality postulates introduced in the previous section. Recall that systems that violate them return undesirable outputs. Before presenting our study, we start first by analyzing the outputs of argumentation systems under the naive semantics. One may wonder why especially since this particular semantics is not used in the literature for evaluating arguments. The reason is that the only case where stable semantics ensures an intuitive output is where an argumentation system returns exactly the same output under stable and naive semantics.

4.1 Naive semantics

Before characterizing the outputs of an AS under naive semantics, let us start by showing some useful properties. The next result shows that if each naive extension returns a consistent subbase of Σ, then the AS is certainly closed under sub-arguments.

Proposition 1 *Let* $\mathcal{T}$ = $(\mathtt{Arg}(\Sigma)$, $\mathcal{R})$ *be an AS over a knowledge base* Σ *such that* $\mathcal{R}$ *is conflict-dependent. If* $\forall \mathcal{E} \in \mathtt{Ext}_n(\mathcal{T})$, $\mathtt{Base}(\mathcal{E})$ *is consistent, then* $\mathcal{T}$ *is closed under sub-arguments (under naive semantics).*

A consequence of the previous result is that under naive semantics, the satisfaction of both consistency and closure under sub-arguments is equivalent to the satisfaction of a stronger version of consistency.

Theorem 1. *Let* $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ *be an AS over a knowledge base* Σ *such that* $\mathcal{R}$ *is conflict-dependent.* $\mathcal{T}$ *satisfies consistency and closure under sub-arguments (under naive semantics) iff* $\forall \mathcal{E} \in \mathtt{Ext}_n(\mathcal{T})$, $\mathtt{Base}(\mathcal{E})$ *is consistent.*

In case of naive semantics, closure under the consequence operator CN is induced from the two other postulates: closure under sub-arguments and consistency.

Proposition 2 *Let* $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ *be an AS over a knowledge base* Σ *such that* $\mathcal{R}$ *is conflict-dependent. If* $\mathcal{T}$ *satisfies consistency and is closed under sub-arguments (under naive semantics), then it is also closed under* CN.

An important question now is: what is hidden behind naive semantics? We show that the naive extensions of *any* argumentation system that satisfies Postulates 2 and 3 *always* return maximal (for set inclusion) consistent subbases of Σ.

Theorem 2. *Let* $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ *be an AS over a knowledge base* Σ *such that* $\mathcal{R}$ *is conflict-dependent. If* $\mathcal{T}$ *satisfies consistency and is closed under sub-arguments (under naive semantics), then:*

- *For all* $\mathcal{E} \in \mathtt{Ext}_n(\mathcal{T})$, $\mathtt{Base}(\mathcal{E}) \in \mathtt{Max}(\Sigma)$.
- *For all* $\mathcal{E}_i, \mathcal{E}_j \in \mathtt{Ext}_n(\mathcal{T})$, *if* $\mathtt{Base}(\mathcal{E}_i) = \mathtt{Base}(\mathcal{E}_j)$ *then* $\mathcal{E}_i = \mathcal{E}_j$.

The previous result does not guarantee that all the maximal consistent subbases of Σ are captured. The next theorem confirms that *any* maximal consistent subbase of Σ defines a naive extension of an AS which satisfies consistency and closure under sub-arguments.

Theorem 3. *Let* $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ *be an AS over a knowledge base* Σ *such that* $\mathcal{R}$ *is conflict-dependent. If* $\mathcal{T}$ *satisfies consistency and is closed under sub-arguments (under naive semantics), then:*

- *For all* $\mathcal{S} \in \mathtt{Max}(\Sigma)$, $\mathtt{Arg}(\mathcal{S}) \in \mathtt{Ext}_n(\mathcal{T})$.
- *For all* $\mathcal{S}_i, \mathcal{S}_j \in \mathtt{Max}(\Sigma)$, *if* $\mathtt{Arg}(\mathcal{S}_i) = \mathtt{Arg}(\mathcal{S}_j)$ *then* $\mathcal{S}_i = \mathcal{S}_j$.

It follows that any argumentation system that satisfies the two postulates 2 and 3 enjoy a full correspondence between the maximal consistent subbases of Σ and the naive extensions of the system.

Theorem 4. *Let* $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ *be an AS over a knowledge base* Σ *such that* $\mathcal{R}$ *is conflict-dependent.* $\mathcal{T}$ *satisfies consistency and is closed under sub-arguments (under naive semantics) iff the naive extensions of* $\mathtt{Ext}_n(\mathcal{T})$ *are exactly the* $\mathtt{Arg}(\mathcal{S})$ *where* $\mathcal{S}$ *ranges over the elements of* $\mathtt{Max}(\Sigma)$.

A direct consequence of the previous result is that the number of naive extensions is finite. This follows naturally from the finiteness of the knowledge base Σ.

Theorem 5. *Let $\mathcal{T}$ = ($\texttt{Arg}(\Sigma)$, $\mathcal{R}$) be an AS over a knowledge base Σ such that $\mathcal{R}$ is conflict-dependent and $\mathcal{T}$ satisfies consistency and is closed under sub-arguments (under naive semantics). If Σ is finite, then $\mathcal{T}$ has a finite number of naive extensions.*

Let us now characterize the set $\texttt{Output}(\mathcal{T})$ of inferences that may be drawn from a knowledge base Σ by an argumentation system $\mathcal{T}$ under naive semantics. It coincides with the set of formulas that are drawn by all the maximal consistent subbases of Σ.

Theorem 6. *Let $\mathcal{T}$ = ($\texttt{Arg}(\Sigma)$, $\mathcal{R}$) be an AS over a knowledge base Σ such that $\mathcal{R}$ is conflict-dependent, $\mathcal{T}$ satisfies consistency and is closed under sub-arguments (under naive semantics).* $\texttt{Output}(\mathcal{T}) = \bigcap \texttt{CN}(\mathcal{S}_i)$ *where* $\mathcal{S}_i \in \texttt{Max}(\Sigma)$.

In short, under naive semantics, *any* 'good' instantiation of Dung's abstract framework returns exactly the formulas that are drawn (with CN) by all the maximal consistent subbases of the base Σ. So whatever the attack relation that is chosen, the result will be the same. It is worth recalling that the output set contains exactly the so-called *universal conclusions* in the approach developed in [14] for reasoning about inconsistent propositional bases.

4.2 Stable semantics

As for naive semantics, we show that under stable semantics, strong consistency induce closure under sub-arguments.

Proposition 3 *Let $\mathcal{T}$ = ($\texttt{Arg}(\Sigma)$, $\mathcal{R}$) be an AS over a knowledge base Σ such that $\mathcal{R}$ is conflict-dependent. If $\forall \mathcal{E} \in \texttt{Ext}_s(\mathcal{T})$, $\texttt{Base}(\mathcal{E})$ is consistent, then $\mathcal{T}$ is closed under sub-arguments (under stable semantics).*

The following theorem shows that satisfying consistency and closure under sub-arguments amounts exactly to satisfying the strong version of consistency.

Theorem 7. *Let $\mathcal{T}$ = ($\texttt{Arg}(\Sigma)$, $\mathcal{R}$) be an AS over a knowledge base Σ such that $\mathcal{R}$ is conflict-dependent. $\mathcal{T}$ satisfies consistency and closure under sub-arguments (under stable semantics) iff $\forall \mathcal{E} \in \texttt{Ext}_s(\mathcal{T})$, $\texttt{Base}(\mathcal{E})$ is consistent.*

Like for naive semantics, in case of stable semantics, closure under the consequence operator CN follows from closure under sub-arguments and consistency.

Proposition 4 *Let $\mathcal{T}$ = ($\texttt{Arg}(\Sigma)$, $\mathcal{R}$) be an AS over a knowledge base Σ such that $\mathcal{R}$ is conflict-dependent. If $\mathcal{T}$ satisfies consistency and is closed under sub-arguments (under stable semantics), then it is also closed under* CN *(under stable semantics).*

We now show that the stable extensions of *any* argumentation system, which satisfies Postulates 2 and 3, return maximal consistent subbases of Σ. This means that if one instantiates Dung's system and does not get maximal consistent subbases with stable extensions, then the instantiation certainly violates one or both of the two key postulates: consistency and closure under sub-arguments.

Theorem 8. *Let $\mathcal{T}$ = ($\texttt{Arg}(\Sigma)$, $\mathcal{R}$) be an AS over a knowledge base Σ such that $\mathcal{R}$ is conflict-dependent. If $\mathcal{T}$ satisfies consistency and closure under sub-arguments (under stable semantics), then:*

- *For all $\mathcal{E} \in \texttt{Ext}_s(\mathcal{T})$, $\texttt{Base}(\mathcal{E}) \in \texttt{Max}(\Sigma)$.*
- *For all $\mathcal{E} \in \texttt{Ext}_s(\mathcal{T})$, $\mathcal{E} = \texttt{Arg}(\texttt{Base}(\mathcal{E}))$.*
- *For all $\mathcal{E}_i, \mathcal{E}_j \in \texttt{Ext}_s(\mathcal{T})$, if $\texttt{Base}(\mathcal{E}_i)$ = $\texttt{Base}(\mathcal{E}_j)$ then $\mathcal{E}_i = \mathcal{E}_j$.*

This result is strong as it characterizes the outputs under stable semantics of a large class of argumentation systems, namely, those grounded on Tarskian logics.

The previous result does not guarantee that each maximal consistent subbase of Σ has a corresponding stable extension in the argumentation system $\mathcal{T}$. To put it differently, it does not guarantee the equality $|\texttt{Ext}_s(\mathcal{T})| = |\texttt{Max}(\Sigma)|$. However, it shows that in case stable extensions exist, then their bases are certainly elements of $\texttt{Max}(\Sigma)$. This enables us to delimit the number of stable extensions that an AS may have.

Proposition 5 *Let $\mathcal{T}$ = ($\texttt{Arg}(\Sigma)$, $\mathcal{R}$) be an AS over a base Σ s.t. $\mathcal{R}$ is conflict-dependent and $\mathcal{T}$ satisfies consistency and closure under sub-arguments (under stable semantics). It holds that $0 \leq |\texttt{Ext}_s(\mathcal{T})| \leq |\texttt{Max}(\Sigma)|$.*

From this property, it follows that when the knowledge base is finite, the number of stable extensions is finite as well.

Property 1 *If Σ is finite, then $\forall \mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ s.t. $\mathcal{T}$ satisfies consistency and closure under sub-arguments (under stable semantics), $|\texttt{Ext}_s(\mathcal{T})|$ is finite.*

The fact that an argumentation system $\mathcal{T}$ verifies or not the equality $|\texttt{Ext}_s(\mathcal{T})| = |\texttt{Max}(\Sigma)|$ depends broadly on the attack relation that is chosen. Let $\Re_s$ be the set of *all* attack relations that ensure Postulates 2 and 3 under stable semantics ($\Re_s = \{\mathcal{R} \subseteq \texttt{Arg}(\Sigma) \times \texttt{Arg}(\Sigma) \mid \mathcal{R}$ is conflict-dependent and $(\texttt{Arg}(\Sigma), \mathcal{R})$ satisfies Postulates 2 and 3 under stable semantics$\}$ for all Σ). This set contains three *disjoints* subsets of attack relations: $\Re_s = \Re_{s1} \cup \Re_{s2} \cup \Re_{s3}$:

- $\Re_{s1}$: the relations which lead to $|\texttt{Ext}_s(\mathcal{T})| = 0$.
- $\Re_{s2}$: the relations which ensure $0 < |\texttt{Ext}_s(\mathcal{T})| < |\texttt{Max}(\Sigma)|$.
- $\Re_{s3}$: the relations which ensure $|\texttt{Ext}_s(\mathcal{T})| = |\texttt{Max}(\Sigma)|$.

Let us analyze separately each category of attack relations. We start with relations of the set $\Re_{s3}$. Those relations induce a one to one correspondence between the stable extensions of the argumentation system and the maximal consistent subbases of Σ.

Property 2 *Let $\mathcal{T}$ = ($\texttt{Arg}(\Sigma)$, $\mathcal{R}$) be an AS over a knowledge base Σ such that $\mathcal{R} \in \Re_{s3}$. For all $\mathcal{S} \in \texttt{Max}(\Sigma)$, $\texttt{Arg}(\mathcal{S}) \in \texttt{Ext}_s(\mathcal{T})$.*

An important question now is: do such attack relations exist? We are not interested in identifying all of them since they lead to the same result. It is sufficient to show whether they exist. Hopefully, such relations exist and *assumption attack* [9] is one of them.

Theorem 9. *The set* $\Re_{s3}$ *is not empty.*

We show now that argumentation systems based on this category of attack relations always have stable extensions.

Theorem 10. *For all* $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ *such that* $\mathcal{R} \in \Re_{s3}$, $\texttt{Ext}_s(\mathcal{T}) \neq \emptyset$.

Let us now characterize the output set of a system under stable semantics.

Theorem 11. *Let* $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ *be an AS over a knowledge base* Σ *such that* $\mathcal{R} \in \Re_{s3}$. $\texttt{Output}(\mathcal{T}) = \bigcap \texttt{CN}(\mathcal{S}_i)$ *where* $\mathcal{S}_i \in \texttt{Max}(\Sigma)$.

Note that this category of attack relations leads exactly to the same result as naive semantics. Thus, stable semantics does not play any particular role. Moreover, argumentation systems return the universal conclusions (of the coherence-based approach [14]) under any monotonic logic, not only under propositional logic as in [14].

Let us now analyze the first category ($\Re_{s1}$) of attack relations that guarantee the postulates. Recall that these relations prevent the existence of stable extensions.

Theorem 12. *Let* $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ *be an AS over a knowledge base* Σ *such that* $\mathcal{R} \in \Re_{s1}$. *It holds that* $\texttt{Output}(\mathcal{T}) = \emptyset$.

These attack relations are skewed, and may prevent intuitive conclusions from being drawn from a knowledge base. This is particularly the case of free formulas, i.e. in $\texttt{Free}(\Sigma)$, as shown next.

Property 3 *Let* $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ *be an AS over a knowledge base* Σ *such that* $\texttt{Free}(\Sigma) \neq \emptyset$. *If* $\mathcal{R} \in \Re_{s1}$, *then* $\forall x \in \texttt{Free}(\Sigma)$, $x \notin \texttt{Output}(\mathcal{T})$.

What about the remaining attack relations, i.e., those of $\Re_{s2}$ that ensure the existence of stable extensions? Systems that use these relations choose a proper subset of maximal consistent subbases of Σ and make inferences from them. Their output sets are defined as follows:

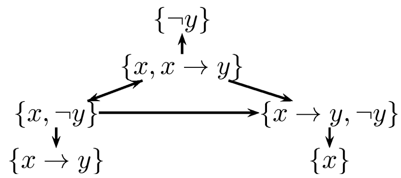
Theorem 13. *Let* $\mathcal{T} = (\texttt{Arg}(\Sigma), \mathcal{R})$ *be an AS over a knowledge base* Σ *such that* $\mathcal{R} \in \Re_{s2}$. $\texttt{Output}(\mathcal{T}) = \bigcap \texttt{CN}(\mathcal{S}_i)$ *where* $\mathcal{S}_i \in \texttt{Max}(\Sigma)$ *and* $\mathcal{S}_i = \texttt{Base}(\mathcal{E}_i)$ *with* $\mathcal{E}_i \in \texttt{Ext}_s(\mathcal{T})$.

These attack relations lead to an unjustified discrimination between the maximal consistent subbases of a knowledge base. Unfortunately, this is fatal for the argumentation systems which use them as they return counter-intuitive results (see Example 1).

Example 1 *Assume that* $(\mathcal{L}, \texttt{CN})$ *is propositional logic and let* Σ *contain three intuitively* equally preferred *formulas:* $\Sigma = \{x, x \to y, \neg y\}$. *This base has three maximal consistent subbases:*

- $\mathcal{S}_1 = \{x, x \to y\}$, $\mathcal{S}_2 = \{x, \neg y\}$, $\mathcal{S}_3 = \{x \to y, \neg y\}$.

The arguments that may be built from Σ may have the following supports: $\{x\}$, $\{x \to y\}$, $\{\neg y\}$, $\{x, x \to y\}$, $\{x, \neg y\}$, and $\{x \to y, \neg y\}$. Assume now the attack relation shown in the figure below. For the sake of readability, we do not represent the conclusions of the arguments in the figure. An arrow from X towards Y is read as follows: any argument with support X attacks any argument with support Y.



This argumentation system has two stable extensions:

- $\mathcal{E}_1 = \{a \in \mathtt{Arg}(\Sigma) \mid \mathtt{Supp}(a) = \{x, x \to y\}$ *or* $\mathtt{Supp}(a) = \{x\}$ *or* $\mathtt{Supp}(a) = \{x \to y\}\}$.
- $\mathcal{E}_2 = \{a \in \mathtt{Arg}(\Sigma) \mid \mathtt{Supp}(a) = \{x, \neg y\}$ *or* $\mathtt{Supp}(a) = \{x\}$ *or* $\mathtt{Supp}(a) = \{\neg y\}\}$.

It can be checked that this argumentation system satisfies consistency and closure under sub-arguments. The two extensions capture respectively the subbases $\mathcal{S}_1$ and $\mathcal{S}_2$.
It is worth noticing that the third subbase $\mathcal{S}_3 = \{x \to y, \neg y\}$ is not captured by any stable extension. Indeed, the set $\mathtt{Arg}(\mathcal{S}_3) = \{a \in \mathtt{Arg}(\Sigma) \mid \mathtt{Supp}(a) = \{\neg y, x \to y\}$ or $\mathtt{Supp}(a) = \{\neg y\}$ or $\mathtt{Supp}(a) = \{x \to y\}\}$ is not a stable extension. $\mathcal{S}_3$ is discarded due to the definition of the attack relation. Note that this leads to non-intuitive outputs. For instance, it can be checked that $x \in \mathtt{Output}(\mathcal{T})$ whereas $\neg y \notin \mathtt{Output}(\mathcal{T})$ and $x \to y \notin \mathtt{Output}(\mathcal{T})$. Since the three formulas of Σ are assumed to be equally preferred, then there is no reason to privilege one compared to the others!

The example showed a skewed attack relation which led to 'artificial' priorities among the formulas of a base Σ: x is preferred to $\neg y$ and $x \to y$. The following result confirms this observation.

Theorem 14. *Let $\mathcal{T}$ = ($\mathtt{Arg}(\Sigma)$, $\mathcal{R}$) be an AS over a knowledge base Σ. If $\mathcal{R} \in \Re_{s2}$, then $\exists x, x' \in \mathtt{Inc}(\Sigma)$ such that $x \in \mathtt{Output}(\mathcal{T})$ and $x' \notin \mathtt{Output}(\mathcal{T})$.*

To sum up, there are three categories of attack relations that ensure the three rationality postulates. Two of them ($\Re_{s1}$ and $\Re_{s2}$) should be avoided as they lead to undesirable results. It is worth mentioning that we are not interested here in identifying those relations. Indeed, if they exist, they certainly lead to bad results and thus, should not be used. The third category of relations ($\Re_{s3}$) leads to "correct" results, but argumentation systems based on them return exactly the same results under naive semantics. Thus, stable semantics does not play any particular role in the logic-based argumentation systems we studied in the paper. Moreover, the outputs of the systems coincide with those of the coherence-based approach [14]. As a consequence, argumentation systems inherit the drawbacks of this approach. Let us illustrate this issue by the following example.

Example 2 *Assume that $(\mathcal{L}, \mathtt{CN})$ is propositional logic and let $\Sigma = \{x, \neg x \wedge y\}$. This base has two maximal consistent subbases:*

– $\mathcal{S}_1 = \{x\}$
– $\mathcal{S}_2 = \{\neg x \wedge y\}$

According to the previous results, any instantiation of Dung's framework falls in one of the following cases:

– *Instantiations that use attack relations in* $\Re_{s1}$ *will lead to* $\texttt{Output}(\mathcal{T}) = \emptyset$*. This result is undesirable since* y *should be inferred from* Σ *since it is not part of the conflict.*
– *Instantiations that use attack relations in* $\Re_{s2}$ *will lead either to* $\texttt{Output}(\mathcal{T}) = \texttt{CN}(\{x\})$ *or to* $\texttt{Output}(\mathcal{T}) = \texttt{CN}(\{\neg x \wedge y\})$*. Both outputs are undesirable since they are unjustified. Why* x *and not* $\neg x$ *and vice versa?*
– *Instantiations that use attack relations in* $\Re_{s3}$ *will lead to* $\texttt{Output}(\mathcal{T}) = \emptyset$*. Like the first case, there is no reason to not conclude* y*.*

5 Related work

This paper investigated the outputs of an argumentation system under stable semantics. There are some works in the literature which are somehow related to our. In [11, 13], the authors studied whether some *particular* argumentation systems satisfy some of the rationality postulates presented in this paper. By particular system, we mean a system that is grounded on a particular logic and/or that uses a specific attack relation. In our paper the objective is different. We assumed abstract argumentation systems that satisfy the desirable postulates, and studied their outputs under stable semantics. Two other works, namely [6] and [3], share this objective. In [6], the author studied one particular system: the one that is grounded on propositional logic and uses the "assumption attack" relation [9]. The results got show that assumption attack belongs to our set $\Re_{s3}$. In [3], these results are generalized to argumentation systems that use the same attack relation but grounded on any Tarskian logic. Our work is more general since it completely abstracts from the attack relation. Moreover, it presents a *complete* view of the outputs under stable semantics.

6 Conclusion

This paper characterized for the first time the outputs (under *stable* semantics) of *any* argumentation system that is grounded on a Tarskian logic and that satisfies very basic rationality postulates. The study is very general since it keeps all the parameters of a system unspecified. Namely, Tarskian logics are abstract and no requirement is imposed on the attack relation except the property of conflict-dependency which is mandatory for ensuring the consistency postulate. We identified the maximum number of stable extensions a system may have. We discussed three possible categories of attack relations that may make a system satisfies the postulates. Two of them lead to counter-intuitive results. Indeed, either ad hoc choices are made or interesting conclusions are forgotten like the free formulas. Argumentation systems based on attack relations of the third category enjoy a one to one correspondence between the stable extensions and the maximal consistent subbases of the knowledge base. Consequently, their outputs are the common conclusions drawn from each maximal consistent subbase. This means that stable semantics does not play any particular role for reasoning with inconsistent information since the same result is returned by naive semantics. Moreover, the argumentation approach is equivalent to the coherence-based one. Consequently, it suffers from, the same drawbacks as this latter.

References

1. L. Amgoud. Postulates for logic-based argumentation systems. In *WL4AI: ECAI Workshop on Weighted Logics for AI*, 2012.
2. L. Amgoud and P. Besnard. Bridging the gap between abstract argumentation systems and logic. In *SUM'09*, pages 12–27, 2009.
3. L. Amgoud and P. Besnard. A formal analysis of logic-based argumentation systems. In *SUM'10*, pages 42–55, 2010.
4. P. Besnard and A. Hunter. A logic-based theory of deductive arguments. *Artificial Intelligence*, 128(1-2):203–235, 2001.
5. M. Caminada and L. Amgoud. On the evaluation of argumentation formalisms. *Artificial Intelligence Journal*, 171 (5-6):286–310, 2007.
6. C. Cayrol. On the relation between argumentation and non-monotonic coherence-based entailment. In *IJCAI'95*, pages 1443–1448, 1995.
7. D2.2. Towards a consensual formal model: inference part. *Deliverable of ASPIC project*, 2004.
8. P. M. Dung. On the acceptability of arguments and its fundamental role in nonmonotonic reasoning, logic programming and n-person games. *Artificial Intelligence Journal*, 77:321–357, 1995.
9. M. Elvang-Gøransson, J. Fox, and P. Krause. Acceptability of arguments as 'logical uncertainty. In *ECSQARU'93*, pages 85–90, 1993.
10. A. García and G. Simari. Defeasible logic programming: an argumentative approach. *Theory and Practice of Logic Programming*, 4:95–138, 2004.
11. N. Gorogiannis and A. Hunter. Instantiating abstract argumentation with classical logic arguments: Postulates and properties. *Artificial Intelligence Journal*, 175(9-10):1479–1497, 2011.
12. J. L. Pollock. How to reason defeasibly. *Artificial Intelligence Journal*, 57:1–42, 1992.
13. H. Prakken. An abstract framework for argumentation with structured arguments. *Journal of Argument and Computation*, 1:93–124, 2010.
14. N. Rescher and R. Manor. On inference from inconsistent premises. *Journal of Theory and decision*, 1:179–219, 1970.
15. A. Tarski. *Logic, Semantics, Metamathematics (E. H. Woodger, editor))*, chapter On Some Fundamental Concepts of Metamathematics. Oxford Uni. Press, 1956.

Appendix

Proof of Proposition 1. Let $\mathcal{T} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an AS over a knowledge base Σ such that $\mathcal{R}$ is conflict-dependent. Assume that $\mathcal{T}$ violates closure under sub-arguments. Thus, $\exists \mathcal{E} \in \mathtt{Ext}_n(\mathcal{T})$ such that $\exists a \in \mathcal{E}$ and $\exists b \in \mathtt{Sub}(a)$ with $b \notin \mathcal{E}$. This means that $\mathcal{E} \cup \{b\}$ is conflicting, i.e. $\exists c \in \mathcal{E}$ such that $b\mathcal{R}c$ or $c\mathcal{R}b$. Since $\mathcal{R}$ is conflict-dependent, then $\texttt{Supp}(b) \cup \texttt{Supp}(c)$ is inconsistent. However, $\texttt{Supp}(b) \subseteq \texttt{Supp}(a) \subseteq \texttt{Base}(\mathcal{E})$ and thus, $\texttt{Supp}(b) \cup \texttt{Supp}(c) \subseteq \texttt{Base}(\mathcal{E})$. This means that $\texttt{Base}(\mathcal{E})$ is inconsistent. This contradicts the assumption. ■

Proof of Theorem 1. Assume that an AS $\mathcal{T}$ satisfies Postulates 2 and 3, then from Proposition 5 (in [1]) it follows that $\forall \mathcal{E} \in \texttt{Ext}_n(\mathcal{T})$, $\texttt{Base}(\mathcal{E})$ is consistent.

Assume now that $\forall \mathcal{E} \in \texttt{Ext}_n(\mathcal{T})$, $\texttt{Base}(\mathcal{E})$ is consistent. Then, $\mathcal{T}$ satisfies consistency (Proposition 4, [1]). Moreover, from Proposition 1, $\mathcal{T}$ is closed under sub-arguments. ■

Proof of Proposition 2. Let $\mathcal{T}$ = ($\texttt{Arg}(\Sigma)$, $\mathcal{R}$) be an AS over a knowledge base Σ such that $\mathcal{R}$ is conflict-dependent. Assume that $\mathcal{T}$ is closed under sub-arguments and satisfies consistency. Assume also that $\mathcal{T}$ violates closure under CN. Thus, $\exists \mathcal{E} \in \texttt{Ext}_n(\mathcal{T})$ such that $\texttt{Concs}(\mathcal{E}) \neq \texttt{CN}(\texttt{Concs}(\mathcal{E}))$. This means that $\exists x \in \texttt{CN}(\texttt{Concs}(\mathcal{E}))$ and $x \notin \texttt{Concs}(\mathcal{E})$. Besides, $\texttt{CN}(\texttt{Concs}(\mathcal{E})) \subseteq \texttt{CN}(\texttt{Base}(\mathcal{E}))$. Thus, $x \in \texttt{CN}(\texttt{Base}(\mathcal{E}))$. Since CN verifies finiteness, then $\exists X \subseteq \texttt{Base}(\mathcal{E})$ such that X is finite and $x \in \texttt{CN}(X)$. Moreover, from Proposition 5 (in [1]), $\texttt{Base}(\mathcal{E})$ is consistent. Then, X is consistent as well (from Property 2 in [2]). Consequently, the pair (X, x) is an argument. Besides, since $x \notin \texttt{Concs}(\mathcal{E})$ then $(X, x) \notin \mathcal{E}$. This means that $\exists a \in \mathcal{E}$ such that $a\mathcal{R}(X, x)$ or $(X, x)\mathcal{R}a$. Finally, since $\mathcal{R}$ is conflict-dependent, then $\texttt{Supp}(a) \cup X$ is inconsistent and consequently $\texttt{Base}(\mathcal{E})$ is inconsistent. This contradicts the assumption. ■

Proof of Theorem 2. Let $\mathcal{T}$ = ($\texttt{Arg}(\Sigma)$, $\mathcal{R}$) be an AS over a knowledge base Σ such that $\mathcal{R}$ is conflict-dependent. Assume that $\mathcal{T}$ satisfies consistency and is closed under sub-arguments. Let $\mathcal{E} \in \texttt{Ext}_n(\mathcal{T})$. From Proposition 5 (in [1]), $\texttt{Base}(\mathcal{E})$ is consistent.

Assume now that $\texttt{Base}(\mathcal{E})$ is not maximal for set inclusion. Thus, $\exists x \in \Sigma \backslash \texttt{Base}(\mathcal{E})$ such that $\texttt{Base}(\mathcal{E}) \cup \{x\}$ is consistent. This means that $\{x\}$ is consistent. Thus, $\exists a \in \texttt{Arg}(\Sigma)$ such that $\texttt{Supp}(a) = \{x\}$. Since $x \notin \texttt{Base}(\mathcal{E})$, then $a \notin \mathcal{E}$. Since $\mathcal{E}$ is a naive extension, then $\exists b \in \mathcal{E}$ such that $a\mathcal{R}b$ or $b\mathcal{R}a$. Since $\mathcal{R}$ is conflict-dependent, then $\texttt{Supp}(a) \cup \texttt{Supp}(b)$ is inconsistent. But, $\texttt{Supp}(b) \subseteq \texttt{Base}(\mathcal{E})$, this would mean that $\texttt{Base}(\mathcal{E}) \cup \{x\}$ is inconsistent. Contradiction.

Let $\mathcal{E} \in \texttt{Ext}_n(\mathcal{T})$. It is obvious that $\mathcal{E} \subseteq \texttt{Arg}(\texttt{Base}(\mathcal{E}))$ since the construction of arguments is monotonic. Let $a \in \texttt{Arg}(\texttt{Base}(\mathcal{E}))$. Thus, $\texttt{Supp}(a) \subseteq \texttt{Base}(\mathcal{E})$. Assume that $a \notin \mathcal{E}$, then $\exists b \in \mathcal{E}$ such that $a\mathcal{R}b$ or $b\mathcal{R}a$. Since $\mathcal{R}$ is conflict-dependent, then $\texttt{Supp}(a) \cup \texttt{Supp}(b)$ is inconsistent. Besides, $\texttt{Supp}(a) \cup \texttt{Supp}(b) \subseteq \texttt{Base}(\mathcal{E})$. This means that $\texttt{Base}(\mathcal{E})$ is inconsistent. Contradiction.

Let now $\mathcal{E}_i, \mathcal{E}_j \in \texttt{Ext}_n(\mathcal{T})$. Assume that $\texttt{Base}(\mathcal{E}_i) = \texttt{Base}(\mathcal{E}_j)$. Then, $\texttt{Arg}(\texttt{Base}(\mathcal{E}_i)) = \texttt{Arg}(\texttt{Base}(\mathcal{E}_j))$. Besides, from the previous bullet, $\mathcal{E}_i = \texttt{Arg}(\texttt{Base}(\mathcal{E}_i))$ and $\mathcal{E}_j = \texttt{Arg}(\texttt{Base}(\mathcal{E}_j))$. Consequently, $\mathcal{E}_i = \mathcal{E}_j$. ■

Proof of Theorem 3. Let $\mathcal{T}$ = ($\texttt{Arg}(\Sigma)$, $\mathcal{R}$) be an AS over a knowledge base Σ s.t. $\mathcal{R}$ is conflict-dependent. Assume that $\mathcal{T}$ satisfies Postulates 2 and 3.

Let $\mathcal{S} \in \texttt{Max}(\Sigma)$, and assume that $\texttt{Arg}(\mathcal{S}) \notin \texttt{Ext}_n(\mathcal{T})$. Since $\mathcal{R}$ is conflict-dependent and $\mathcal{S}$ is consistent, then it follows from Proposition 5 in [2] that $\texttt{Arg}(\mathcal{S})$ is conflict-free. Thus, $\texttt{Arg}(\mathcal{S})$ is not maximal for set inclusion. So, $\exists a \in \texttt{Arg}(\Sigma)$ such that $\texttt{Arg}(\mathcal{S}) \cup \{a\}$ is conflict-free. There are two possibilities: i) $\mathcal{S} \cup \texttt{Supp}(a)$ is consistent. But since $\mathcal{S} \in \texttt{Max}(\Sigma)$, then $\texttt{Supp}(a) \subseteq \mathcal{S}$, and this would mean that $a \in \texttt{Arg}(\mathcal{S})$. ii) $\mathcal{S} \cup \texttt{Supp}(a)$

is inconsistent. Thus, $\exists C \in \mathcal{C}_\Sigma$ such that $C \subseteq \mathcal{S} \cup \mathtt{Supp}(a)$. Let $X_1 = C \cap \mathcal{S}$ and $X_2 = C \cap \mathtt{Supp}(a)$. From Lemma 3 in [1], $\exists x_1 \in \mathtt{CN}(X_1)$ and $\exists x_2 \in \mathtt{CN}(X_2)$ such that the set $\{x_1, x_2\}$ is inconsistent. Note that (X_1, x_1) and (X_2, x_2) are arguments. Moreover, $(X_1, x_1) \in \mathtt{Arg}(\mathcal{S})$ and $(X_2, x_2) \in \mathtt{Sub}(a)$. Besides, since $\mathtt{Arg}(\mathcal{S}) \cup \{a\}$ is conflict-free, then $\exists \mathcal{E} \in \mathtt{Ext}(\mathcal{T})$ such that $\mathtt{Arg}(\mathcal{S}) \cup \{a\} \subseteq \mathcal{E}$. Thus, $(X_1, x_1) \in \mathcal{E}$. Since $\mathcal{T}$ is closed under sub-arguments then $(X_2, x_2) \in \mathcal{E}$. Thus, $\{x_1, x_2\} \subseteq \mathtt{Concs}(\mathcal{E})$. From Property 2 in [2], it follows that $\mathtt{Concs}(\mathcal{E})$ is inconsistent. This contradicts the fact that $\mathcal{T}$ satisfies consistency.

Let now $\mathcal{S}_i, \mathcal{S}_j \in \mathtt{Max}(\Sigma)$ be such that $\mathtt{Arg}(\mathcal{S}_i) = \mathtt{Arg}(\mathcal{S}_j)$. Assume that $\mathcal{S}_i \neq \mathcal{S}_j$, thus $\exists x \in \mathcal{S}_i$ and $x \notin \mathcal{S}_j$. Besides, $\mathcal{S}_i$ is consistent, then so is the set $\{x\}$. Consequently, $\exists a \in \mathtt{Arg}(\Sigma)$ such that $\mathtt{Supp}(a) = \{x\}$. It follows also that $a \in \mathtt{Arg}(\mathcal{S}_i)$ and thus $a \in \mathtt{Arg}(\mathcal{S}_j)$. By definition of an argument, $\mathtt{Supp}(a) \subseteq \mathcal{S}_j$. Contradiction. ∎

Proof of Theorem 4. Let $\mathcal{T}$ = ($\mathtt{Arg}(\Sigma)$, $\mathcal{R}$) be an AS over a knowledge base Σ such that $\mathcal{R}$ is conflict-dependent.

Assume that $\mathcal{T}$ satisfies Postulates 2 and 3. Them from Theorems 2 and 3, it follows that there is a full correspondence between $\mathtt{Max}(\Sigma)$ and $\mathtt{Ext}_n(\mathcal{T})$.

Assume now that there is a full correspondence between $\mathtt{Max}(\Sigma)$ and $\mathtt{Ext}_n(\mathcal{T})$. Then, $\forall \mathcal{E} \in \mathtt{Ext}(\mathcal{T})$, $\mathtt{Base}(\mathcal{E})$ is consistent. Consequently, $\mathcal{T}$ satisfies consistency. Moreover, from Proposition 1, $\mathcal{T}$ is closed under sub-arguments. ∎

Proof of Theorem 5. Let $\mathcal{T}$ = ($\mathtt{Arg}(\Sigma)$, $\mathcal{R}$) be an AS over a knowledge base Σ such that $\mathcal{R}$ is conflict-dependent and $\mathcal{T}$ satisfies consistency and is closed under sub-arguments. From Theorem 4, it follows that $|\mathtt{Ext}_n(\mathcal{T})| = |\mathtt{Max}(\Sigma)|$. Since Σ is finite, then it has a finite number of maximal consistent subbases. Thus, the number of naive extensions is finite as well. ∎

Proof of Theorem 6. Let $\mathcal{T}$ = ($\mathtt{Arg}(\Sigma)$, $\mathcal{R}$) be an AS over a knowledge base Σ such that $\mathcal{R}$ is conflict-dependent. Assume that $\mathcal{T}$ satisfies consistency and is closed under sub-arguments. Then, from Proposition 2, $\mathcal{T}$ enjoys closure under $\mathtt{CN}$. Then, from Property 5 in [1], for all $\mathcal{E} \in \mathtt{Ext}_n(\mathcal{T})$, $\mathtt{Concs}(\mathcal{E}) = \mathtt{CN}(\mathtt{Base}(\mathcal{E}))$. Finally, from Theorem 4, there is a full correspondence between elements of $\mathtt{Max}(\Sigma)$ and the naive extensions. Thus, for all $\mathcal{E}_i \in \mathtt{Ext}_n(\mathcal{T})$, $\exists! \mathcal{S}_i \in \mathtt{Max}(\Sigma)$ such that $\mathtt{Base}(\mathcal{E}_i) = \mathcal{S}_i$. Thus, $\mathtt{Concs}(\mathcal{E}_i) = \mathtt{CN}(\mathcal{S}_i)$. By definition, $\mathtt{Output}(\mathcal{T}) = \bigcap \mathtt{Concs}(\mathcal{E}_i)$, thus $\mathtt{Output}(\mathcal{T}) = \bigcap \mathtt{CN}(\mathcal{S}_i)$. ∎

Proof of Proposition 3. Let $\mathcal{T}$ = ($\mathtt{Arg}(\Sigma)$, $\mathcal{R}$) be an AS over a knowledge base Σ such that $\mathcal{R}$ is conflict-dependent. Assume that $\forall \mathcal{E} \in \mathtt{Ext}_s(\mathcal{T})$, $\mathtt{Base}(\mathcal{E})$ is consistent. Assume also that $\mathcal{T}$ violates closure under sub-arguments. Thus, $\exists \mathcal{E} \in \mathtt{Ext}_s(\mathcal{T})$ such that $\exists a \in \mathcal{E}$ and $\exists b \in \mathtt{Sub}(a)$ with $b \notin \mathcal{E}$. Since $\mathcal{E}$ is a stable extension, then $\exists c \in \mathcal{E}$ such that $c\mathcal{R}b$. Since $\mathcal{R}$ is conflict-dependent, then $\mathtt{Supp}(b) \cup \mathtt{Supp}(c)$ is inconsistent. However, $\mathtt{Supp}(b) \subseteq \mathtt{Supp}(a) \subseteq \mathtt{Base}(\mathcal{E})$. Then, $\mathtt{Supp}(b) \cup \mathtt{Supp}(c) \subseteq \mathtt{Base}(\mathcal{E})$. This means that $\mathtt{Base}(\mathcal{E})$ is inconsistent. This contradicts the assumption. ∎

Proof of Theorem 7. Assume that an AS $\mathcal{T}$ satisfies Postulates 2 and 3, then from Proposition 5 (in [1]) it follows that $\forall \mathcal{E} \in \mathtt{Ext}_s(\mathcal{T})$, $\mathtt{Base}(\mathcal{E})$ is consistent.

Assume now that $\forall \mathcal{E} \in \mathtt{Ext}_s(\mathcal{T})$, $\mathtt{Base}(\mathcal{E})$ is consistent. Then, $\mathcal{T}$ satisfies consistency (Proposition 4, [1]). Moreover, from Proposition 3, $\mathcal{T}$ is closed under sub-arguments. ∎

Proof of Proposition 4. Let $\mathcal{T}$ = ($\texttt{Arg}(\Sigma)$, $\mathcal{R}$) be an AS over a knowledge base Σ such that $\mathcal{R}$ is conflict-dependent. Assume that $\mathcal{T}$ is closed under sub-arguments and satisfies consistency. Assume also that $\mathcal{T}$ violates closure under $\texttt{CN}$. Thus, $\exists \mathcal{E} \in \texttt{Ext}_s(\mathcal{T})$ such that $\texttt{Concs}(\mathcal{E}) \neq \texttt{CN}(\texttt{Concs}(\mathcal{E}))$. This means that $\exists x \in \texttt{CN}(\texttt{Concs}(\mathcal{E}))$ and $x \notin \texttt{Concs}(\mathcal{E})$. Besides, $\texttt{CN}(\texttt{Concs}(\mathcal{E})) \subseteq \texttt{CN}(\texttt{Base}(\mathcal{E}))$. Thus, $x \in \texttt{CN}(\texttt{Base}(\mathcal{E}))$. Since $\texttt{CN}$ verifies finiteness, then $\exists X \subseteq \texttt{Base}(\mathcal{E})$ such that X is finite and $x \in \texttt{CN}(X)$. Moreover, from Proposition 5 (in [1]), $\texttt{Base}(\mathcal{E})$ is consistent. Then, X is consistent as well (from Property 2 in [2]). Consequently, the pair (X, x) is an argument. Besides, since $x \notin \texttt{Concs}(\mathcal{E})$ then $(X, x) \notin \mathcal{E}$. This means that $\exists a \in \mathcal{E}$ such that $a\mathcal{R}(X, x)$. Finally, since $\mathcal{R}$ is conflict-dependent, then $\texttt{Supp}(a) \cup X$ is inconsistent and consequently $\texttt{Base}(\mathcal{E})$ is inconsistent. This contradicts the assumption. ■

Proof of Theorem 8. Let $\mathcal{T}$ = ($\texttt{Arg}(\Sigma)$, $\mathcal{R}$) be an AS over a knowledge base Σ such that $\mathcal{R}$ is conflict-dependent. Let $\mathcal{E} \in \texttt{Ext}_s(\mathcal{T})$. Since $\mathcal{T}$ satisfies Postulates 1, 2 and 3, then $\texttt{Base}(\mathcal{E})$ is consistent (from Proposition 3). Assume now that $\texttt{Base}(\mathcal{E})$ is not maximal for set inclusion. Thus, $\exists x \in \Sigma \setminus \texttt{Base}(\mathcal{E})$ such that $\texttt{Base}(\mathcal{E}) \cup \{x\}$ is consistent. This means that $\{x\}$ is consistent. Thus, $\exists a \in \texttt{Arg}(\Sigma)$ such that $\texttt{Supp}(a) = \{x\}$. Since $x \notin \texttt{Base}(\mathcal{E})$, then $a \notin \mathcal{E}$. Since $\mathcal{E}$ is a stable extension, then $\exists b \in \mathcal{E}$ such that $b\mathcal{R}a$. Since $\mathcal{R}$ is conflict-dependent, then $\texttt{Supp}(a) \cup \texttt{Supp}(b)$ is inconsistent. But, $\texttt{Supp}(b) \subseteq \texttt{Base}(\mathcal{E})$, this would mean that $\texttt{Base}(\mathcal{E}) \cup \{x\}$ is inconsistent. Contradiction.

Let $\mathcal{E} \in \texttt{Ext}_s(\mathcal{T})$. It is obvious that $\mathcal{E} \subseteq \texttt{Arg}(\texttt{Base}(\mathcal{E}))$ since the construction of arguments is monotonic. Let $a \in \texttt{Arg}(\texttt{Base}(\mathcal{E}))$. Thus, $\texttt{Supp}(a) \subseteq \texttt{Base}(\mathcal{E})$. Assume that $a \notin \mathcal{E}$, then $\exists b \in \mathcal{E}$ such that $b\mathcal{R}a$. Since $\mathcal{R}$ is conflict-dependent, then $\texttt{Supp}(a) \cup \texttt{Supp}(b)$ is inconsistent. Besides, $\texttt{Supp}(a) \cup \texttt{Supp}(b) \subseteq \texttt{Base}(\mathcal{E})$. This means that $\texttt{Base}(\mathcal{E})$ is inconsistent. Contradiction.

Let now $\mathcal{E}_i, \mathcal{E}_j \in \texttt{Ext}_s(\mathcal{T})$. Assume that $\texttt{Base}(\mathcal{E}_i) = \texttt{Base}(\mathcal{E}_j)$. Then, $\texttt{Arg}(\texttt{Base}(\mathcal{E}_i)) = \texttt{Arg}(\texttt{Base}(\mathcal{E}_j))$. Besides, from bullet 2 of this proof, $\mathcal{E}_i = \texttt{Arg}(\texttt{Base}(\mathcal{E}_i))$ and $\mathcal{E}_j = \texttt{Arg}(\texttt{Base}(\mathcal{E}_j))$. Consequently, $\mathcal{E}_i = \mathcal{E}_j$. ■

Proof of Proposition 5. Let $\mathcal{T}$ = ($\texttt{Arg}(\Sigma)$, $\mathcal{R}$) be an AS over a knowledge base Σ s.t. $\mathcal{R}$ is conflict-dependent and $\mathcal{T}$ satisfies consistency and closure under sub-arguments. If $\texttt{Ext}_s(\mathcal{T}) = \emptyset$, then $|\texttt{Ext}_s(\mathcal{T})| = 0$. If $\texttt{Ext}_s(\mathcal{T}) \neq \emptyset$, then $|\texttt{Ext}_s(\mathcal{T})| \leq |\texttt{Max}(\Sigma)|$ (from Theorem 8). ■

Proof of Property 2. Let $\mathcal{T}$ = ($\texttt{Arg}(\Sigma)$, $\mathcal{R}$) be an AS over a knowledge base Σ such that $\mathcal{R} \in \Re_{s3}$. Let $\mathcal{S} \in \texttt{Max}(\Sigma)$. Since $|\texttt{Ext}(\mathcal{T})| = |\texttt{Max}(\Sigma)|$, then from Theorem 8, $\exists \mathcal{E} \in \texttt{Ext}_s(\mathcal{T})$ such that $\texttt{Base}(\mathcal{E}) = \mathcal{S}$. Besides, from the same theorem, $\mathcal{E} = \texttt{Arg}(\texttt{Base}(\mathcal{E}))$, thus $\mathcal{E} = \texttt{Arg}(\mathcal{S})$. Consequently, $\texttt{Arg}(\mathcal{S}) \in \texttt{Ext}_s(\mathcal{T})$. ■

Proof of Theorem 9. It was shown under any Tarskian logic that the attack relation proposed in [9] and called *assumption attack* verifies the correspondence between stable extensions and maximal subbases. Thus, assumption attack belongs to $\Re_{s3}$. ■

Proof of Theorem 10. Let $\mathcal{T}$ = ($\texttt{Arg}(\Sigma)$, $\mathcal{R}$) be an AS over a knowledge base Σ such that $\mathcal{R} \in \Re_{s3}$. Then, $|\texttt{Ext}_s(\mathcal{T})| = |\texttt{Max}(\Sigma)|$. There are two cases: i) Σ contains only inconsistent formulas, thus $\texttt{Max}(\Sigma) = \{\emptyset\}$ and $\texttt{Ext}_s(\mathcal{T}) = \{\emptyset\}$ since $\texttt{Arg}(\Sigma) = \emptyset$. ii) Σ contains at lest one consistent formula x. Thus, $\exists \mathcal{S} \in \texttt{Max}(\Sigma)$ such $x \in \mathcal{S}$. Since $\mathcal{R} \in \Re_{s3}$, then $\texttt{Arg}(\mathcal{S}) \in \texttt{Ext}_s(\mathcal{T})$. ■