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Equivalence in logic-based argumentation

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This paper investigates when two abstract logic-based argumentation systems are *equivalent*. It defines various equivalence criteria, investigates the links between them, and identifies cases where two systems are equivalent with respect to each of the proposed criteria. In particular, it shows that under some reasonable conditions on the logic underlying an argumentation system, the latter has an equivalent *finite* subsystem, called *core*. This core constitutes a threshold under which arguments of the system have not yet attained their final status and consequently adding a new argument may result in status change. From that threshold, the statuses of all arguments become stable.

Keywords: argumentation; equivalence; logic

1. Introduction

Argumentation is a reasoning process in which interacting arguments are built and evaluated. It is widely studied in Artificial Intelligence, namely for reasoning about inconsistent information (Bondarenko, Dung, Kowalski, & Toni, 1997; Garcia & Simari, 2004; Governatori, Maher, Antoniou, & Billington, 2004), making decisions (Kakas & Moraitis, 2003; Labreuche, 2006; Amgoud & Prade, 2009), and modeling agents interactions (Reed, 1998; Amgoud, Maudet, & Parsons, 2000; Prakken, 2006).

One of the most abstract argumentation systems was proposed by Dung (1995). It consists of a *set of arguments* and a binary relation representing *conflicts* among them. Several *semantics* were proposed by the same author and by others for evaluating the arguments (Baroni, Giacomin, & Guida, 2005; Caminada, 2006b; Dung, Mancarella, & Toni, 2007). Each of them consists of a set of criteria that should be satisfied by any acceptable set of arguments, called *extension*. From the extensions, a *status* is assigned to each argument: An argument is *sceptically* accepted if it appears in each extension, it is *credulously* accepted if it belongs to some extensions and not to others, and finally it is *rejected* if it is not in any extension. Several key decision problems were identified (like whether an argument is sceptically accepted under a given semantics), and their

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computational complexity investigated (Dunne, 2007; Dunne & Wooldridge, 2009). Most of the results concern *finite* argumentation systems (i.e., systems that have finite sets of arguments).

Almost all existing argumentation systems for reasoning about inconsistent information are instantiations of this abstract system - except Delp system developed Garcia and Simari (2004) and the one proposed by Besnard and Hunter (2008). An instantiation starts with a logic $(\mathcal{L}, \mathrm{CN})$ where $\mathcal{L}$ is the language of this logic and CN its consequence operator. It considers as input a knowledge base whose formulae are elements of the language $\mathcal{L}$. From this base, arguments and attacks among them are defined using the consequence operator CN. Finally, a semantics is chosen for evaluating the arguments. Examples of such logic-based systems are those based on propositional logic (Cayrol, 1995; Amgoud & Cayrol, 2002; Gorogiannis & Hunter, 2011). It is worth noticing that most logics induce an infinite number of arguments from a given knowledge base. This is unfortunately true even when the knowledge base itself is finite which means that such systems cannot benefit from existing results on finite systems and makes it hard to apply them. An important question is then: is it possible to find a finite sub-system of an infinite one (called target) that is able to compute all the outputs of the target? This amounts to checking whether the finite sub-system (if any) is *equivalent* to its target.

Equivalence is a key notion in several domains. In logic it defines interchangeable formulae. It can, for instance, be used to identify knowledge bases that have the same sets of models. The notion of equivalence has also gained interest in the area of knowledge representation (Lifschitz, Pearce, & Valverde, 2001). In general, the idea is to see when two objects / systems / programs, ... are not the same but have the same behavior. More recently, several works have been done on equivalence in argumentation, namely equivalence of Dung's style systems.

Oikarinen and Woltran (2011) distinguished two kinds of equivalence: *basic* or *standard equivalence* and *strong equivalence*. For each of them, they proposed three equivalence criteria. Two systems are *basically equivalent* if they have the same extensions (resp. the same sets of sceptically/credulously accepted arguments). They are strongly equivalent if their expansions with any arbitrary argumentation system have the same extensions (resp. the same sets of sceptically/credulously accepted arguments). Baumann (2012) proposed four forms of equivalence: *normal expansion equivalence*, *strong expansion equivalence*, *weak expansion equivalence* and *local expansion equivalence*. The basic idea behind the four forms is to put restrictions on the kind of systems that expand the two original ones, i.e., the two that are compared. For instance, with strong expansion equivalence, a system can only be expanded by a system whose arguments are never attacked by the arguments of the former. It was shown that the four forms are "between" basic and strong equivalence. More links, under various semantics, between the six forms of equivalence were established by Baumann and Brewka (2013). Another work which somehow tackled the notion of equivalence in argumentation theory was done by Baroni et al. (2012). The authors defined input/output argumentation systems which characterise the behavior of systems under Dung's semantics. They have shown that systems having the same behavior can be interchanged, meaning that they are equivalent.

A common theme in all works mentioned above is the use of abstract argumentation systems (i.e., systems where neither the origin nor the structure of arguments and attacks are known). None of those proposals consider equivalence of structured or instantiated argumentation systems. Consequently, the different notions of equivalence may be poor since they do not consider the contents of arguments and are thus syntax-dependent. Moreover, the existing notions of equivalence, except the basic one, are more appropriate in dynamic contexts where the set of arguments of a given system may evolve. This is, for instance the case in dialogs where new arguments and thus new attacks may be received. However, in a reasoning context, the set of arguments of a system is static and it is built from a given knowledge base. In such a setting, one may (for example) want to check whether two systems built from two different knowledge bases are equivalent. One may also want to check whether the system can be replaced by an equivalent sub-system in order to reduce the computation bulk.

The goal of this paper is to study when two reasoning systems, i.e., instantiated argumentation systems are equivalent. We do not focus neither on a particular logic nor on a particular attack relation. We rather study abstract but structured argumentation systems. Indeed, we assume systems that are built under the abstract monotonic logics of Tarski (1956) and that use *any* attack relation. We start by extending the list of equivalence criteria by new ones. These latter consider outputs which are proper to logic-based argumentation systems, like the plausible inferences. We show that those criteria are too rigid since they do not take into account the structure of arguments and are syntax-dependent. We then refine them using a new notion of equivalence of arguments and the classical notion of equivalence of formulae. We investigate the links between the different criteria. Some of the results hold for any attack relation while others hold only when the attack relations of the two systems enjoy some intuitive properties. We then identify cases where two systems are equivalent with respect to each of the proposed criteria.

Another contribution of the paper consists of showing that under some reasonable conditions on the logic $(\mathcal{L}, \mathrm{CN})$, each argumentation system has an equivalent *finite* subsystem, called *core*. The core is seen as the smallest sub-system that retrieves all the outputs of its target. This notion is of great importance not only for replacing infinite systems by finite ones, but also for replacing finite systems by smaller ones. Indeed, it is well-known that building arguments from a knowledge base is computationally a complex task. Consider the case of a propositional base. An argument is usually defined as a logical proof containing a consistent subset of the base, called support, and a given statement, called conclusion. Thus, there are at least two tests to be done: a consistency test which is an NP-complete problem and an inference test (i.e., testing whether the conclusion is a logical consequence of the support) which is a co-NP-complete problem. Hence, finding the components of an argumentation system is a real challenge. Exchanging a system with its core may thus considerably reduce the bulk of computation.

The paper is organized as follows: In Section 2, we recall the logic-based argumentation systems we are interested in. In Section 3, we study equivalence. We propose various equivalence criteria, study their inter-dependencies and provide conditions under which two systems are equivalent with respect to each of the proposed criteria. Section 4 defines the notion of a core of an argumentation system, studies when a core is finite, and investigates its role in dynamic situations. The last section is devoted to some concluding remarks and perspectives. All the proofs are put in an appendix.

2. Logic-Based Argumentation Systems

This section describes the logic-based argumentation systems we are interested in. They are built around the abstract monotonic logic proposed by Tarski (1956). Such a logic is a pair $(\mathcal{L}, \mathrm{CN})$ where $\mathcal{L}$ is any set of *well-formed formulae* and CN is a *consequence* *operator*, i.e., a function from $2^{\mathcal{L}}$ to $2^{\mathcal{L}}$ that satisfies the following five postulates:

- $X \subseteq \mathrm{CN}(X)$ (Expansion)
- $\mathrm{CN}(\mathrm{CN}(X)) = \mathrm{CN}(X)$ (Idempotence)
- $\mathrm{CN}(X) = \bigcup_{Y \subseteq_f X} \mathrm{CN}(Y)$[1] (Finiteness)
- $\mathrm{CN}(\{x\}) = \mathcal{L}$ for some $x \in \mathcal{L}$ (Absurdity)
- $\mathrm{CN}(\emptyset) \neq \mathcal{L}$ (Coherence)

Intuitively, $\mathrm{CN}(X)$ returns the set of formulae that are logical consequences of X according to the logic at hand. Almost all well-known logics (classical logic, intuitionistic logic, modal logics, ...) are special cases of Tarski's notion of monotonic logic. In such a logic, the notion of consistency is defined as follows.

Definition 1 (Consistency)**.** *A set $X \subseteq \mathcal{L}$ is* consistent *iff* $\mathrm{CN}(X) \neq \mathcal{L}$. *It is* inconsistent *otherwise.*

Arguments are built from a *knowledge base* Σ, a finite subset of $\mathcal{L}$. They are minimal (for set inclusion) proofs for some statements, called their conclusions.

Definition 2 (Argument)**.** *Let* $(\mathcal{L}, \mathrm{CN})$ *be a Tarskian logic and* $\Sigma \subseteq \mathcal{L}$. *An* argument *built from* Σ *is a pair* (X, x) *s.t.*

- *X is a finite consistent subset of Σ*
- *$x \in \mathcal{L}$,*
- *$x \in \mathrm{CN}(X)$,*
- *$\nexists X' \subset X$ s.t. $x \in \mathrm{CN}(X')$.*

X is the support *of the argument and x its* conclusion.

The following example illustrates the previous definition.

Example 1. *Let* $(\mathcal{L}, \mathrm{CN})$ *be propositional logic (a Tarskian logic) and* $\Sigma = \{x, \neg y, x \rightarrow y\}$ *be a knowledge base. Examples of arguments that may be built from this base are:*

- $(\{x\}, x)$, $(\{\neg y\}, \neg y)$, $(\{x \rightarrow y\}, x \rightarrow y)$
- $(\{x, x \rightarrow y\}, y)$, $(\{x, \neg y\}, x \wedge \neg y)$, $(\{\neg y, x \rightarrow y\}, \neg x)$
- $(\{x\}, x \wedge x)$, $(\{x\}, x \vee y)$, $(\{x\}, x \vee z)$
- ...

The previous definition specified what we accept as an argument. It is worth mentioning that the set of *all* arguments that may be built from a knowledge base may be infinite even when the base is itself finite. This depends on the underlying logic. This is, for instance, the case under propositional logic. Thus, this is also the case in the previous example.

Notations: For an argument $a = (X, x)$, $\mathrm{Conc}(a) = x$ and $\mathrm{Supp}(a) = X$. For a set $\mathcal{S} \subseteq \mathcal{L}$, $\mathrm{Arg}(\mathcal{S}) = \{a \mid a$ is an argument (in the sense of Definition 2) and $\mathrm{Supp}(a) \subseteq \mathcal{S}\}$. For any set $\mathcal{E} \subseteq \mathrm{Arg}(\mathcal{L})$ of arguments, $\mathrm{Base}(\mathcal{E}) = \bigcup_{a \in \mathcal{E}} \mathrm{Supp}(a)$.

An attack relation $\mathcal{R}$ is defined on a given set $\mathcal{A}$ of arguments, i.e., $\mathcal{R} \subseteq \mathcal{A} \times \mathcal{A}$. The writing $a\mathcal{R}b$ (or $(a, b) \in \mathcal{R}$) means that the argument a attacks the argument b. This relation expresses disagreements between arguments. Amgoud and Besnard (2009) argue that it should capture the inconsistency of the knowledge base. An example of such a relation is the so-called *assumption attack* relation (Elvang-Gøransson, Fox, &

[1]The notation $Y \subseteq_f X$ means that Y is a finite subset of X.

Krause, 1993). According to this relation, an argument attacks another if it undermines one of the formulae of its support. In the sequel, the attack relation is left unspecified.

A logic-based instantiation of Dung's argumentation system is defined as follows.

Definition 3 (Argumentation system). *An* argumentation system *built over a knowledge base* Σ *is a pair* $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ *where* $\mathcal{A} \subseteq \text{Arg}(\Sigma)$ *and* $\mathcal{R} \subseteq \mathcal{A} \times \mathcal{A}$ *is an attack relation.*

Almost all existing argumentation systems consider the whole set $\text{Arg}(\Sigma)$ of arguments. For the purpose of this paper, we do not need to make this assumption. The reason is that we are looking for equivalent systems, thus, we may be interested in a sub-system which is equivalent to the 'complete' system (i.e., the one with the whole set $\text{Arg}(\Sigma)$ of arguments). We may also need to compare two sub-systems of a given complete system. When the set of arguments is infinite, then the corresponding argumentation system is said to be infinite.

Definition 4 (Finite argumentation system). *An argumentation system* $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ *is* finite *iff the set* $\mathcal{A}$ *is finite. It is* infinite *otherwise.*

In what follows, arguments are evaluated using the semantics proposed by Dung (1995). Before recalling them, let us first introduce the two requirements on which they are based: *conflict-freeness* and *defence*.

Definition 5 (Conflict-freeness – Defence). *Let* $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ *be an argumentation system,* $\mathcal{E} \subseteq \mathcal{A}$ *and* $a \in \mathcal{A}$.

- $\mathcal{E}$ *is* conflict-free *iff* $\nexists a, b \in \mathcal{E}$ *s.t.* $a \mathcal{R} b$
- $\mathcal{E}$ defends a *iff* $\forall b \in \mathcal{A}$, *if* $b \mathcal{R} a$ *then* $\exists c \in \mathcal{E}$ *s.t.* $c \mathcal{R} b$.

The next definition introduces the different semantics we are considering in this paper. Note that there are several other semantics in the literature like *semi-stable* semantics (Caminada, 2006b), *ideal* semantics (Dung et al., 2007), and the *recursive* ones (Baroni et al., 2005). However, for the purpose of this paper, we do not need to recall of them. The main aim of the paper is to formalize the concept of equivalence in argumentation, and to show how it can be used for different purposes. The ideas hold under any semantics. Thus, we choose the most common ones.

Definition 6 (Acceptability semantics). *Let* $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ *be an argumentation system and* $\mathcal{E} \subseteq \mathcal{A}$. *We say that* $\mathcal{E}$ *is* admissible *iff it is conflict-free and defends all its elements.*

- $\mathcal{E}$ *is a* complete *extension iff it is admissible and contains any argument it defends.*
- $\mathcal{E}$ *is a* preferred *extension iff it is a maximal (for set inclusion) admissible set.*
- $\mathcal{E}$ *is a* stable *extension iff it is conflict-free and* $\forall a \in \mathcal{A} \setminus \mathcal{E}$, $\exists b \in \mathcal{E}$ *s.t.* $b \mathcal{R} a$.
- $\mathcal{E}$ *is a* grounded extension *iff it is a minimal (for set inclusion) complete extension.*

Let $\text{Ext}_x(\mathcal{F})$ *denote the set of all extensions of the argumentation system* $\mathcal{F}$ *under semantics* x *where* $x \in \{c, p, s, g\}$ *and* c *(resp.* p, s, g*) stands for complete (resp. preferred, stable and grounded). When we do not need to refer to a particular semantics, we use the notation* $\text{Ext}(\mathcal{F})$ *for short.*

Throughout the paper, we use the term "all reviewed semantics" to refer to the four semantics stated in the previous definition (i.e. complete, preferred, stable and grounded semantics). When a result is stated without referring to a particular semantics, it means that it holds for all the reviewed semantics. It is worth recalling that grounded semantics guarantees one extension while all the other semantics may ensure several extensions. Note also that in general, an argumentation system may have an infinite number of extensions even if the knowledge base Σ is finite. Let us consider the following example.

Example 2. *Let* $(\mathcal{L}, \mathrm{CN})$ *be a Tarski's logic such that the set* $\mathcal{L}$ *contains an infinite number of formulae,* $\mathcal{L} = \{x_0, x_1, x_2, \ldots\}$*) and*

$$\mathrm{CN}(X) = \begin{cases} \emptyset & \text{if } X = \emptyset \\ \{x_i, x_{i+1}, x_{i+2}, \ldots\} & \text{else, where } i \text{ is the minimal number s.t. } x_i \in X \end{cases}$$

Consider now the knowledge base $\Sigma = \{x_1\}$ *and the attack relation defined as follows:*

For two arguments a and b, $a\mathcal{R}b$ *iff* $\mathrm{Conc}(a) \neq \mathrm{Conc}(b)$

The argumentation system $(\mathrm{Arg}(\Sigma), \mathcal{R})$ *has an infinite number of stable extensions:* $\{(\{x_1\}, x_1)\}$, $\{(\{x_1\}, x_2)\}$, $\{(\{x_1\}, x_3)\}, \ldots$.

An extension (under a given semantics) represents a coherent position or point of view. Thus, it contains arguments that may hold all together. However, the status of a given argument is determined with respect to all the extensions. An argument is either 1) *sceptically* accepted (if it belongs to all the extensions), or 2) *credulously* accepted (if it belongs to some but not all extensions), or 3) *rejected* if (it does not belong to any extension).

Definition 7 (Status of arguments)**.** *Let* $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ *be an argumentation system and* $a \in \mathcal{A}$.

- *a is* sceptically accepted *iff* $a \in \bigcap_{\mathcal{E}_i \in \mathrm{Ext}(\mathcal{F})} \mathcal{E}_i$
- *a is* credulously accepted *iff* $a \in \bigcup_{\mathcal{E}_i \in \mathrm{Ext}(\mathcal{F})} \mathcal{E}_i$
- *a is* rejected *iff* $a \notin \bigcup_{\mathcal{E}_i \in \mathrm{Ext}(\mathcal{F})} \mathcal{E}_i$

Let $\mathrm{Status}(a, \mathcal{F})$ *be a function which returns the status of argument a in system* $\mathcal{F}$.

The following definition summarizes all the possible outputs of an argumentation system.

Definition 8 (Outputs)**.** *Let* $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ *be an argumentation system built over a knowledge base* Σ.

- $\mathrm{Ext}(\mathcal{F})$ *is the set of extensions of* $\mathcal{F}$ *under a given semantics*
- $\mathrm{Sc}(\mathcal{F}) = \{a \in \mathcal{A} \mid a \text{ is sceptically accepted } \}$
- $\mathrm{Cr}(\mathcal{F}) = \{a \in \mathcal{A} \mid a \text{ is credulously accepted } \}$
- $\mathrm{Output}_{sc}(\mathcal{F}) = \{\mathrm{Conc}(a) \mid a \text{ is sceptically accepted}\}$
- $\mathrm{Output}_{cr}(\mathcal{F}) = \{\mathrm{Conc}(a) \mid a \text{ is credulously accepted}\}$
- $\mathrm{Bases}(\mathcal{F}) = \{\mathrm{Base}(\mathcal{E}) \mid \mathcal{E} \in \mathrm{Ext}(\mathcal{F})\}$

The first set contains the extensions of a system $\mathcal{F}$ under a given semantics. The four next sets contain the sceptically and credulously accepted arguments (resp. conclusions). The set $\mathrm{Bases}(\mathcal{F})$ contains the subbases of Σ which are computed by the extensions of $\mathcal{F}$. Note that the three last outputs can only be defined for structured argumentation systems. Finally, it is worth noticing that all the five last outputs follow from the extensions.

3. Equivalence

The notion of equivalence in argumentation theory is of great importance since it defines which systems are interchangeable. This is crucial for comparing systems using different attack relations, or for replacing a system by a smaller one.

3.1 *Equivalence criteria*

We assume a *fixed* Tarskian logic $(\mathcal{L}, \mathrm{CN})$. This means that we study the equivalence of two systems that are grounded on the *same logic*. This assumption is not strong since:

(1) the kind of applications in which equivalence is needed assume that the two systems to be compared use the same logic, and
(2) it is difficult to compare different logics since they may have different expressive power.

We consider two arbitrary argumentation systems $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ and $\mathcal{F}' = (\mathcal{A}', \mathcal{R}')$ that are defined using the fixed logic. Note that the two systems may be built over different knowledge bases (respectively Σ and Σ').

The study of equivalence of two argumentation systems passes through the definition of equivalence *criteria*. We propose two families of criteria. Both compare the outputs of the two systems. However, the first family is syntax-dependent while the second family takes advantage of similarities between arguments (respectively formulae). The following definition introduces the criteria of the first family. Recall that the first three criteria were already proposed by Oikarinen and Woltran (2011).

Definition 9 (Equivalence criteria)**.** *Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ and $\mathcal{F}' = (\mathcal{A}', \mathcal{R}')$ be two argumentation systems built using the same Tarskian logic $(\mathcal{L}, \mathrm{CN})$. $\mathcal{F}$ and $\mathcal{F}'$ are* equivalent with respect to criterion EQi, *denoted by $\mathcal{F} \equiv_{EQi} \mathcal{F}'$, iff EQi holds where $i \in \{1, \ldots, 6\}$ and:*

EQ1 $\mathrm{Ext}(\mathcal{F}) = \mathrm{Ext}(\mathcal{F}')$
EQ2 $\mathrm{Sc}(\mathcal{F}) = \mathrm{Sc}(\mathcal{F}')$
EQ3 $\mathrm{Cr}(\mathcal{F}) = \mathrm{Cr}(\mathcal{F}')$
EQ4 $\mathrm{Output}_{sc}(\mathcal{F}) = \mathrm{Output}_{sc}(\mathcal{F}')$
EQ5 $\mathrm{Output}_{cr}(\mathcal{F}) = \mathrm{Output}_{cr}(\mathcal{F}')$
EQ6 $\mathrm{Bases}(\mathcal{F}) = \mathrm{Bases}(\mathcal{F}')$.

The three first criteria concern arguments whereas the three others refer to formulae. For instance, criterion $EQ1$ ensures that the two argumentation systems have exactly the same extensions (under a given semantics) whereas criterion $EQ4$ compares the conclusions that are drawn from the knowledge bases of the two systems. Note that rejected arguments are not considered when comparing two argumentation systems. Indeed, the set of rejected arguments is not an important output of a system (compared to sceptical and credulous arguments). Moreover, it is exactly the complement of the set of credulous arguments. Let us consider the following example.

Example 3. *Let $(\mathcal{L}, \mathrm{CN})$ be propositional logic. Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ and $\mathcal{F}' = (\mathcal{A}', \mathcal{R}')$ be two argumentation systems such that:*

- $\mathcal{A} = \{a_1, a_2\}$ *and* $\mathcal{R} = \{(a_1, a_2)\}$
- $\mathcal{A}' = \{a_2, a_3\}$ *and* $\mathcal{R}' = \{(a_3, a_2)\}$

with:

- $a_1 = (\{t \wedge \neg x\}, \neg x)$,
- $a_2 = (\{x, y\}, x \wedge y)$,
- $a_3 = (\{w \wedge \neg y\}, \neg y)$.

Under grounded (resp. complete, preferred, stable) semantics, $\text{Ext}(\mathcal{F}) = \{\{a_1\}\}$ *and* $\text{Ext}(\mathcal{F}') = \{\{a_3\}\}$. *It is easy to see that* $\mathcal{F}$ *and* $\mathcal{F}'$ *would be equivalent if we compare rejected arguments since their sets of rejected arguments coincide (i.e., the set* $\{a_2\}$*). However, the two systems have almost nothing in common since neither their conclusions (*$\neg x$ *resp.* $\neg y$*) nor their arguments coincide.*

The previous criteria do not take into account the possible similarities/equivalences between arguments or between formulae. Consequently, they are too rigid and may miss some clear equivalences between argumentation systems as illustrated by the following example.

Example 4. *Let* $(\mathcal{L}, \text{CN})$ *be propositional logic. Let us consider two argumentation systems* $\mathcal{F}$ *and* $\mathcal{F}'$ *such that* $\text{Ext}(\mathcal{F}) = \{\mathcal{E}\}$, $\text{Ext}(\mathcal{F}') = \{\mathcal{E}'\}$ *and*

- $\mathcal{E} = \{(\{x \to y\}, x \to y)\}$,
- $\mathcal{E}' = \{(\{x \to y\}, \neg x \vee y)\}$.

The two systems $\mathcal{F}$ *and* $\mathcal{F}'$ *are equivalent with respect to criterion EQ6 since* $\text{Bases}(\mathcal{F}) = \text{Bases}(\mathcal{F}') = \{\{x \to y\}\}$. *However, they are not equivalent with respect to the remaining criteria since the two arguments* $(\{x \to y\}, x \to y)$ *and* $(\{x \to y\}, \neg x \vee y)$ *(resp. the two formulae* $x \to y$ *and* $\neg x \vee y$*) are considered as different.*

This example shows that the six criteria are *syntax-dependent.* Indeed, they consider the two arguments $(\{x \to y\}, x \to y)$ and $(\{x \to y\}, \neg x \vee y)$ as different even if they have the same supports and logically equivalent conclusions. Let us now consider a different example which shows another limit of the previous criteria.

Example 5. *Let* $(\mathcal{L}, \text{CN})$ *be propositional logic. Let us consider two argumentation systems* $\mathcal{F}$ *and* $\mathcal{F}'$ *such that* $\text{Ext}(\mathcal{F}) = \{\mathcal{E}\}$, $\text{Ext}(\mathcal{F}') = \{\mathcal{E}'\}$ *and*

- $\mathcal{E} = \{(\{x, \neg\neg y\}, x \wedge y)\}$ *and*
- $\mathcal{E}' = \{(\{x, y\}, x \wedge y)\}$.

The two systems $\mathcal{F}$ *and* $\mathcal{F}'$ *are equivalent with respect to EQ4 and EQ5 but are not equivalent with respect to the remaining criteria, including EQ6. However, for each formula in* $\text{Bases}(\mathcal{F}) = \{\{x, \neg\neg y\}\}$, *there is an equivalent one in* $\text{Bases}(\mathcal{F}) = \{\{x, y\}\}$ *and vice versa.*

The two previous examples show that in order to have more refined equivalence criteria, the logical equivalence between formulae and between sets of formulae should be considered.

Definition 10 (Equivalence of formulae)**.** *Let* $x, y \in \mathcal{L}$ *and* $X, Y \subseteq \mathcal{L}$.

- *The two formulae* x *and* y *are equivalent, denoted by* $x \equiv y$, *iff* $\text{CN}(\{x\}) = \text{CN}(\{y\})$. *We write* $x \not\equiv y$ *otherwise.*
- X *and* Y *are equivalent, denoted by* $X \cong Y$, *iff* $\forall x \in X$, $\exists y \in Y$ *s.t.* $x \equiv y$ *and* $\forall y \in Y, \exists x \in X$ *s.t.* $x \equiv y$. *We write* $X \ncong Y$ *otherwise.*

Example 6. *In case of propositional logic, the two sets* $\{x, \neg\neg y\}$ *and* $\{x, y\}$ *from Example 5 are equivalent.*

Note that if $X \cong Y$, then $\mathrm{CN}(X) = \mathrm{CN}(Y)$. However, the converse is not true. For instance, $\mathrm{CN}(\{x \wedge y\}) = \mathrm{CN}(\{x, y\})$ while $\{x \wedge y\} \ncong \{x, y\}$. One may ask why not to use the equality of $\mathrm{CN}(X)$ and $\mathrm{CN}(Y)$ in order to say that X and Y are equivalent? The previous example might have already given some of our motivation for such a definition: wanting to make a distinction between $\{x, y\}$ and $\{x \wedge y\}$. The following example of two argumentation systems whose credulous conclusions are respectively $\{x, \neg x\}$ and $\{y, \neg y\}$ is more drastic: it is clear that $\mathrm{CN}(\{x, \neg x\}) = \mathrm{CN}(\{y, \neg y\})$ while the two sets are in no way similar.

In order to define an accurate notion of equivalence between two argumentation systems, we also take advantage of equivalence of arguments. There are three ways of defining such equivalence as shown in the next definition.

Definition 11 (Equivalence of arguments)**.** *Let* $a, a' \in \mathrm{Arg}(\mathcal{L})$*.*

- $a \approx_1 a'$ *iff* $\mathrm{Supp}(a) = \mathrm{Supp}(a')$ *and* $\mathrm{Conc}(a) \equiv \mathrm{Conc}(a')$
- $a \approx_2 a'$ *iff* $\mathrm{Supp}(a) \cong \mathrm{Supp}(a')$ *and* $\mathrm{Conc}(a) = \mathrm{Conc}(a')$
- $a \approx_3 a'$ *iff* $\mathrm{Supp}(a) \cong \mathrm{Supp}(a')$ *and* $\mathrm{Conc}(a) \equiv \mathrm{Conc}(a')$

Example 7. *The two arguments* $(\{x \to y\}, x \to y)$ *and* $(\{x \to y\}, \neg x \vee y)$ *From Example 4 are equivalent with respect to criteria* $\approx_1$ *and* $\approx_3$*.*

Note that each criterion $\approx_i$ is an *equivalence relation* (i.e. reflexive, symmetric and transitive).

Property 1. *Each criterion* $\approx_i$ *is an equivalence relation (with* $i \in \{1, 2, 3\}$*).*

The following property summarizes the links between the three criteria and shows that criterion $\approx_3$ is more general than the two others.

Property 2. *Let* $a, a' \in \mathrm{Arg}(\mathcal{L})$*.*

- *If* $a \approx_1 a'$*, then* $a \approx_3 a'$
- *If* $a \approx_2 a'$*, then* $a \approx_3 a'$

It is worth mentioning that two argumentation systems may have arguments that are equivalent with respect to $\approx_1$ and other arguments that are equivalent with respect to $\approx_2$. Thus, none of the two criteria ($\approx_1, \approx_2$) is able to capture both equivalences. However, criterion $\approx_3$ does. Thus, for the purpose of our paper, we will consider criterion $\approx_3$. Throughout the paper, we refer to this criterion by $\approx$ for short.

The notion of equivalence of two arguments is extended to an equivalence of sets of arguments as follows.

Definition 12 (Equivalence of sets of arguments)**.** *Let* $\mathcal{E}, \mathcal{E}' \subseteq \mathrm{Arg}(\mathcal{L})$*. The two sets* $\mathcal{E}$ *and* $\mathcal{E}'$ *are equivalent, denoted by* $\mathcal{E} \sim \mathcal{E}'$*, iff* $\forall a \in \mathcal{E}, \exists a' \in \mathcal{E}'$ *s.t.* $a \approx a'$ *and* $\forall a' \in \mathcal{E}', \exists a \in \mathcal{E}$ *s.t.* $a \approx a'$*.*

Example 8. *The two extensions* $\{(\{x \to y\}, x \to y)\}$ *and* $\{(\{x \to y\}, \neg x \vee y)\}$ *from Example 4 are equivalent.*

We are now ready to introduce the family of refined equivalence criteria.

Definition 13 (Refined equivalence criteria)**.** *Let* $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ *and* $\mathcal{F}' = (\mathcal{A}', \mathcal{R}')$ *be two argumentation systems built using the same Tarskian logic.* $\mathcal{F}$ *and* $\mathcal{F}'$ *are* equivalent with respect to criterion EQib*, denoted by* $\mathcal{F} \equiv_{EQib} \mathcal{F}'$*, iff EQib holds where* $i = 1 - 6$ *and:*

EQi/EQj	EQ1	EQ1b	EQ2	EQ2b	EQ3	EQ3b	EQ4	EQ4b	EQ5	EQ5b	EQ6	EQ6b
EQ1	+	+	+	+	+	+	+	+	+	+	+	+
EQ1b		+				+				+		+
EQ2			+	+			+	+				
EQ2b				+				+				
EQ3					+	+			+	+		
EQ3b						+				+		
EQ4							+	+				
EQ4b								+				
EQ5									+	+		
EQ5b										+		
EQ6											+	+
EQ6b												+

Table 1. *Links between criteria under any of the reviewed semantics*

EQ1b *there exists a bijection* $f : \mathrm{Ext}(\mathcal{F}) \to \mathrm{Ext}(\mathcal{F}')$ *s.t.* $\forall \mathcal{E} \in \mathrm{Ext}(\mathcal{F})$, $\mathcal{E} \sim f(\mathcal{E})$

EQ2b $\mathrm{Sc}(\mathcal{F}) \sim \mathrm{Sc}(\mathcal{F}')$

EQ3b $\mathrm{Cr}(\mathcal{F}) \sim \mathrm{Cr}(\mathcal{F}')$

EQ4b $\mathrm{Output}_{sc}(\mathcal{F}) \cong \mathrm{Output}_{sc}(\mathcal{F}')$

EQ5b $\mathrm{Output}_{cr}(\mathcal{F}) \cong \mathrm{Output}_{cr}(\mathcal{F}')$

EQ6b $\forall S \in \mathrm{Bases}(\mathcal{F})$, $\exists S' \in \mathrm{Bases}(\mathcal{F}')$ *s.t.* $S \cong S'$ *and* $\forall S' \in \mathrm{Bases}(\mathcal{F}')$, $\exists S \in \mathrm{Bases}(\mathcal{F})$ *s.t.* $S \cong S'$.

Example 9. *The two argumentation systems* $\mathcal{F}$ *and* $\mathcal{F}'$ *from Example 4 are equivalent with respect to the six refined criteria.*

Example 10. *The two argumentation systems* $\mathcal{F}$ *and* $\mathcal{F}'$ *from Example 5 are equivalent with respect to the six refined criteria.*

It is easy to check that each criterion *EQib* refines its strong version *EQi*.

Property 3. *For two argumentation systems* $\mathcal{F}$ *and* $\mathcal{F}'$, *if* $\mathcal{F} \equiv_{EQi} \mathcal{F}'$ *then* $\mathcal{F} \equiv_{EQib} \mathcal{F}'$ *with* $i \in \{1, \dots, 6\}$.

Finally, we show that each of the twelve criteria is an *equivalence relation.*

Property 4. *For all* $i \in \{1, \dots, 6\}$, *the criterion EQi (resp. EQib) is an equivalence relation.*

3.2 *Links between criteria*

In the previous section, we proposed twelve equivalence criteria between argumentation systems. The following result establishes the dependencies between them.

Theorem 1. *Let* $\mathcal{F}$ *and* $\mathcal{F}'$ *be two argumentation systems built on the same logic* $(\mathcal{L}, \mathrm{CN})$. *Table 1 summarises the dependencies* $(\mathcal{F} \equiv_x \mathcal{F}') \Rightarrow (\mathcal{F} \equiv_{x'} \mathcal{F}')$ *under any of the reviewed semantics.*

Table 1 is read as follows: for two criteria, c in row i and c' in column j, the sign + in the intersection of row i and column j means that if two systems are equivalent with respect to c then they are equivalent with respect to c'. For example, the sign + in the intersection of row corresponding to *EQ*1*b* and the column corresponding to *EQ*3*b* means that if two argumentation systems are equivalent with respect to *EQ*1*b* then they must be equivalent with respect to *EQ*3*b*. It is worth noticing that two argumentation systems that are equivalent with respect to *EQ*1 are also equivalent with respect to any of the remaining criteria. This is not the case for its refined version *EQ*1*b*. For instance, two systems that are equivalent with respect to *EQ*1*b* are not necessarily equivalent with respect to $EQ2b$ and $EQ4b$. Thus, $EQ1$ is the most general criterion. This is not surprising since the extensions of a system are at the heart of all the other outputs of an argumentation system. However, as seen in the previous section, the criterion $EQ1$ is too rigid since it does not take into account the internal structure of arguments.

Note that Theorem 1 is a full characterisation in the sense that no other links exist between criteria. In other words, if there is no sign + in Table 1, then it is not the case that the criterion in the corresponding row implies the criterion in the corresponding column. Note some dependencies that might look expected at the first sight but that do not hold in the general case. Given the huge number of cases, we do not provide counter-examples for all of them, since the paper would become unbearably long. The next two examples serve as counter examples for several cases and we strongly believe that the reader can construct counter examples for other missing dependencies.

The next example shows that EQ1b does not imply EQ1, EQ2, EQ3, EQ4, EQ5 nor EQ6 (in the general case). Even more interestingly, from this example we see that EQ1 does not imply neither EQ2b nor EQ4b.

Example 11. *Suppose stable semantics and let* $\mathcal{L} = \{r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8, r_9, r_{10}, c\}$ *with* CN *defined as follows: for all* $X \subseteq \mathcal{L}$,

$$\mathrm{CN}(X) = \begin{cases} \mathcal{L} \setminus \{c\}, & \text{if } c \notin X \text{ and } X \neq \emptyset \\ \mathcal{L}, & \text{if } c \in X \\ \emptyset, & \text{if } X = \emptyset \end{cases}$$

Let $a_1 = (\{r_1\}, r_2)$, $a_2 = (\{r_3\}, r_4)$, $a_3 = (\{r_5\}, r_6)$, $a_4 = (\{r_7\}, r_8)$, $a_5 = (\{r_9\}, r_{10})$. *Let* $\mathcal{A} = \{a_1, a_2, a_3\}$, $\mathcal{R} = \{(a_2, a_3), (a_3, a_2)\}$, $\mathcal{A}' = \{a_4, a_5\}$ *and* $\mathcal{R}' = \{(a_4, a_5), (a_5, a_4)\}$. $\mathrm{Sc}(\mathcal{F}) = \{a_1\}$, $\mathrm{Sc}(\mathcal{F}') = \emptyset$. $\mathcal{F} \equiv_{EQ1b} \mathcal{F}'$ *since a bijection verifying conditions of EQ11 can be defined as:* $f : \mathrm{Ext}(\mathcal{F}) \to \mathrm{Ext}(\mathcal{F}')$, $f(\{a_1, a_2\}) = \{a_4\}$, $f(\{a_1, a_3\}) = \{a_5\}$. *However, criteria like EQ2b and EQ4b are not satisfied.*

We can also show that EQ4 does not imply EQ1, EQ1b, EQ2, EQ2b, EQ3, 3Q3b, EQ6, EQ6b, as illustrated by the following example.

Example 12. *Suppose stable semantics, let* $(\mathcal{L}, \mathrm{CN})$ *be propositional logic and let* $\mathcal{A} = \{(\{x \wedge y\}, x)\}$, $\mathcal{A}' = \{(\{x \wedge z\}, x)\}$, $\mathcal{R} = \emptyset$, $\mathcal{R}' = \emptyset$. $\mathrm{Output}_{sc}(\mathcal{F}) = \mathrm{Output}_{sc}(\mathcal{F}') = \{x\}$.

The previous links between the criteria hold under *all* the acceptability semantics from Definition 6. Somehow expectedly, there are more links between criteria for single-extension semantics, i.e., grounded semantics.

Theorem 2. *The links between the twelve equivalence criteria under grounded semantics are summarized in Table 2.*

The previous results hold for any pair of argumentation systems that are grounded on the same Tarskian logic and whatever the attack relations that are used by the systems. Like in the case of Theorem 1, this result is also "complete" in the sense that no other links exist except those depicted in Table 2.

In what follows, we show that there are additional links between some criteria when the attack relations of the two systems enjoy some properties, namely those discussed by Gorogianis and Hunter (Gorogiannis & Hunter, 2011). Below we recall the ones that are important for our study.

$C1$ $\quad \forall a, b, c \in \mathcal{A}$, if $\mathrm{Conc}(a) = \mathrm{Conc}(b)$ then $(a\mathcal{R}c \Leftrightarrow b\mathcal{R}c)$

$C1b$ $\quad \forall a, b, c \in \mathcal{A}$, if $\mathrm{Conc}(a) \equiv \mathrm{Conc}(b)$ then $(a\mathcal{R}c \Leftrightarrow b\mathcal{R}c)$

EQi/EQj	EQ1	EQ1b	EQ2	EQ2b	EQ3	EQ3b	EQ4	EQ4b	EQ5	EQ5b	EQ6	EQ6b
EQ1	+	+	+	+	+	+	+	+	+	+	+	+
EQ1b		+		+		+		+		+		+
EQ2	+	+	+	+	+	+	+	+	+	+	+	+
EQ2b		+		+		+		+		+		+
EQ3	+	+	+	+	+	+	+	+	+	+	+	+
EQ3b		+		+		+		+		+		+
EQ4							+	+	+	+		
EQ4b								+		+		
EQ5							+	+	+	+		
EQ5b								+		+		
EQ6											+	+
EQ6b												+

Table 2. *Links between criteria under grounded semantics.*

$C2$ $\quad \forall a, b, c \in \mathcal{A}$, if $\text{Supp}(a) = \text{Supp}(b)$ then $(c\mathcal{R}a \Leftrightarrow c\mathcal{R}b)$

$C2b$ $\quad \forall a, b, c \in \mathcal{A}$, if $\text{Supp}(a) \cong \text{Supp}(b)$ then $(c\mathcal{R}a \Leftrightarrow c\mathcal{R}b)$

The two first properties say that two arguments that have the same (resp. equivalent) conclusions attack the same arguments. The two remaining properties say that arguments that have the same (resp. equivalent) supports are attacked by the same arguments. The following result establishes some links between the four properties.

Property 5. *Let $\mathcal{R}$ be an attack relation.*

- *If $\mathcal{R}$ satisfies $C1b$ then it satisfies $C1$.*
- *If $\mathcal{R}$ satisfies $C2b$ then it satisfies $C2$.*

Before presenting the new links, let us first study how the equivalence relation $\approx$ between arguments is related to an attack relation which enjoys the two properties $C1b$ and $C2b$. We show that equivalent arguments (with respect to $\approx$) behave in the same way with respect to attacks in case the attack relation enjoys these two properties.

Property 6. *Let $(\mathcal{A}, \mathcal{R})$ be an argumentation system s.t. $\mathcal{R}$ enjoys $C1b$ and $C2b$. For all $a, a', b, b' \in \mathcal{A}$, $(a \approx a'$ and $b \approx b')$ implies $(a\mathcal{R}b$ iff $a'\mathcal{R}b')$.*

The next result shows that equivalent arguments belong to the same extensions in an argumentation system whose attack relation satisfies $C1b$ and $C2b$.

Property 7. *Let $(\mathcal{A}, \mathcal{R})$ be an argumentation system s.t. $\mathcal{R}$ enjoys $C2b$. For all $a, a' \in \mathcal{A}$, if $a \approx a'$, then $\forall \mathcal{E} \in \text{Ext}(\mathcal{F})$, $a \in \mathcal{E}$ iff $a' \in \mathcal{E}$.*

An obvious consequence of this property is that equivalent arguments have the same status in any argumentation system.

Property 8. *Let $(\mathcal{A}, \mathcal{R})$ be an argumentation system s.t. $\mathcal{R}$ enjoys $C1b$ and $C2b$. For all $a, a' \in \mathcal{A}$, if $a \approx a'$, then $\text{Status}(a, \mathcal{F}) = \text{Status}(a', \mathcal{F})$.*

We also show that two equivalent arguments that belong to two equivalent argumentation systems with respect to criterion $EQ1b$ have the same status.

Property 9. *Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$, $\mathcal{F}' = (\mathcal{A}', \mathcal{R}')$ be two argumentation systems built from the same logic $(\mathcal{L}, \text{CN})$ such that $\mathcal{R}$ and $\mathcal{R}'$ enjoy $C1b$ and $C2b$. If $\mathcal{F} \equiv_{EQ1b} \mathcal{F}'$, then for all $a \in \mathcal{A}$ and for all $a' \in \mathcal{A}'$, if $a \approx a'$ then $\text{Status}(a, \mathcal{F}) = \text{Status}(a', \mathcal{F}')$.*

Finally, we show that if two argumentation systems whose attack relations enjoy $C1b$ and $C2b$ are equivalent with respect to $EQ1b$, then they are also equivalent with respect to $EQ2b$ and $EQ4b$.

Theorem 3. *Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$, $\mathcal{F}' = (\mathcal{A}', \mathcal{R}')$ be two argumentation systems built from the same logic $(\mathcal{L}, \text{CN})$, $\mathcal{R}$ and $\mathcal{R}'$ enjoy $C1b$ and $C2b$. If $\mathcal{F} \equiv_{EQ1b} \mathcal{F}'$, then $\mathcal{F} \equiv_x \mathcal{F}'$*

with $x \in \{EQ2b, EQ4b\}$.

3.3 *Cases of equivalent argumentation systems*

We previously proposed different equivalence criteria of two argumentation systems built from the same logic. An important question now is: *"Are there* distinct *argumentation systems which are equivalent with respect to those criteria?"*. Recall that in case of the criteria proposed by Oikarinen and Woltran (2011) (i.e., $EQ1$, $EQ2$ and $EQ3$), the answer is negative. Indeed, the authors have shown that when two argumentation systems do not have self-attacking arguments, then they are equivalent if an only if they coincide. Amgoud and Besnard (2009) have shown that logic-based argumentation systems do not have self-attacking arguments. This means that the previous criteria are not useful in this context. In what follows, we show that their refinements make it possible to compare different systems. We focus on the criterion $EQ1b$ since it is at the same time general like EQ1 but much more flexible (since syntax-independant).

We start by showing that under some reasonable conditions on the attack relation, an argumentation system built from a knowledge base Σ has a finite number of extensions even if its set of arguments is itself infinite.

Theorem 4. *Let $(\mathcal{A}, \mathcal{R})$ be an argumentation system built over Σ. If Σ is finite and $\mathcal{R}$ satisfies $C2$, then $(\mathcal{A}, \mathcal{R})$ has a finite number of extensions under all reviewed semantics.*

We are now interested in the case of two argumentation systems that may be built from two distinct knowledge bases but use the same attack relation. For instance, both systems use 'rebut' relation or both systems use 'assumption attack', etc. Recall that $\text{Arg}(\mathcal{L})$ is the set of all arguments that can be built from a fixed logical language $\mathcal{L}$ using a fixed consequence operator CN. We denote by $\mathcal{R}_\mathcal{L}$ the attack relation which is used in the two systems with $\mathcal{R}_\mathcal{L} \subseteq \text{Arg}(\mathcal{L}) \times \text{Arg}(\mathcal{L})$. The following result shows under which conditions two systems are equivalent with respect to $EQ1b$.

Theorem 5. *Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ and $\mathcal{F}' = (\mathcal{A}', \mathcal{R}')$ be two argumentation systems s.t. $\mathcal{A}, \mathcal{A}' \subseteq \text{Arg}(\mathcal{L})$ and $\mathcal{R} = \mathcal{R}_\mathcal{L}|_\mathcal{A}, \mathcal{R}' = \mathcal{R}_\mathcal{L}|_{\mathcal{A}'}$. If $\mathcal{R}_\mathcal{L}$ satisfies $C1b$ and $C2b$ and $\mathcal{A} \sim \mathcal{A}'$, then $\mathcal{F} \equiv_{EQ1b} \mathcal{F}'$.*

The following corollary follows from the links between the criteria.

Corollary 6. *Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ and $\mathcal{F}' = (\mathcal{A}', \mathcal{R}')$ be two argumentation systems s.t. $\mathcal{A}, \mathcal{A}' \subseteq \text{Arg}(\mathcal{L})$ and $\mathcal{R} = \mathcal{R}_\mathcal{L}|_\mathcal{A}, \mathcal{R}' = \mathcal{R}_\mathcal{L}|_{\mathcal{A}'}$. If $\mathcal{R}_\mathcal{L}$ satisfies $C1b$ and $C2b$ and $\mathcal{A} \sim \mathcal{A}'$, then $\mathcal{F} \equiv_x \mathcal{F}'$ with $x \in \{EQ2b, EQ3b, EQ4b, EQ5b, EQ6b\}$.*

4. Core(s) of an argumentation system

In this section, we introduce a new concept: *core* of an argumentation system. It is a proper sub-system of an argumentation system which considers *only* one argument among equivalent ones.

Notation: For an arbitrary set X, an arbitrary equivalence relation $\sim$ on X, and $x \in X$, $[x] = \{x' \in X \mid x' \sim x\}$ and $X/\sim\, = \{[x] \mid x \in X\}$. For any $X \subseteq \mathcal{L}$, $\text{Cncs}(X) = \{x \in \mathcal{L} \mid \exists Y \subseteq X \text{ such that } \text{CN}(Y) \neq \mathcal{L} \text{ and } x \in \text{CN}(Y)\}$. In other words, $\text{Cncs}(X)$ is the set of formulae that are drawn from consistent subsets of X.

We define a core as follows.

Definition 14 (Core of an argumentation system)**.** *Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ and $\mathcal{F}' = (\mathcal{A}', \mathcal{R}')$ be two argumentation systems. $\mathcal{F}'$ is a* core *of $\mathcal{F}$ iff:*

- $\mathcal{A}' \subseteq \mathcal{A}$
- $\forall C \in \mathcal{A}/\approx$, $\exists! a \in C \cap \mathcal{A}'$
- $\mathcal{R}' = \mathcal{R}|_{\mathcal{A}'}$ *(i.e., $\mathcal{R}'$ is the restriction of $\mathcal{R}$ on $\mathcal{A}'$).*

It is worth noticing that an argumentation system may have several cores. The set of arguments of each of them is equivalent to the set of arguments of the original system.

Property 10. *Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ and $\mathcal{F}' = (\mathcal{A}', \mathcal{R}')$ be two argumentation systems. If $\mathcal{F}'$ is a core of $\mathcal{F}$, then $\mathcal{A} \sim \mathcal{A}'$.*

When the attack relation enjoys some intuitive properties, an argumentation system is equivalent to any of its cores.

Theorem 7. *Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ and $\mathcal{F}' = (\mathcal{A}', \mathcal{R}')$ be two argumentation systems s.t. $\mathcal{R}$ and $\mathcal{R}'$ satisfy $C1b$ and $C2b$. If $\mathcal{F}'$ is a core of $\mathcal{F}$, then $\mathcal{F} \equiv_{EQ1b} \mathcal{F}'$.*

It follows that the outputs of an argumentation system coincide with those of its cores.

Corollary 8. *Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ and $\mathcal{F}' = (\mathcal{A}', \mathcal{R}')$ be two argumentation systems s.t. $\mathcal{R}$ and $\mathcal{R}'$ satisfy $C1b$ and $C2b$. If $\mathcal{F}'$ is a core of $\mathcal{F}$, then*

- $\mathrm{Sc}(\mathcal{F}) \sim \mathrm{Sc}(\mathcal{F}')$
- $\mathrm{Cr}(\mathcal{F}) \sim \mathrm{Cr}(\mathcal{F}')$
- $\mathrm{Output}_{sc}(\mathcal{F}) \cong \mathrm{Output}_{sc}(\mathcal{F}')$
- $\mathrm{Output}_{cr}(\mathcal{F}) \cong \mathrm{Output}_{cr}(\mathcal{F}')$
- $\mathrm{Bases}(\mathcal{F}) = \mathrm{Bases}(\mathcal{F}')$

A core is seen as a compact version of an argumentation system. The statuses of its arguments are those computed in the original system. Moreover, it is easy to show that each argument which does not belong to a core has an equivalent argument with the same status in the original system.

Property 11. *Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ be an argumentation system and $\mathcal{F}' = (\mathcal{A}', \mathcal{R}')$ its core. If $\mathcal{R}$ satisfies $C1b$ and $C2b$ then:*

- *If $a \in \mathcal{A}'$, then* $\mathrm{Status}(a, \mathcal{F}) = \mathrm{Status}(a, \mathcal{F}')$,
- *If $a \notin \mathcal{A}'$, then* $\mathrm{Status}(a, \mathcal{F}) = \mathrm{Status}(b, \mathcal{F}')$ *for some $b \in \mathcal{A}'$ with $a \approx b$.*

It is worth noticing that the cores of a given argumentation system are equivalent. This follows from the fact that the equivalence criteria (e.g., $EQ1b$) are equivalence relations, thus transitive. So, if $\mathcal{F}$ is an argumentation system and $\mathcal{F}'$ and $\mathcal{F}''$ its cores, then from $\mathcal{F} \equiv_{EQ1b} \mathcal{F}'$ and $\mathcal{F} \equiv_{EQ1b} \mathcal{F}''$, we have $\mathcal{F}' \equiv_{EQ1b} \mathcal{F}''$.

Property 12. *Let $\mathcal{F}'$ and $\mathcal{F}''$ be two cores of an argumentation system $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ such that $\mathcal{R}$ satisfies $C1b$ and $C2b$. It holds that $\mathcal{F}' \equiv_{EQ1b} \mathcal{F}''$.*

We have shown so far how to define a proper sub-system of an argumentation system which is able to compute all the outputs of this later. However, there is no guarantee that the sub-system is finite (i.e., it has a finite set of arguments). In fact, the finiteness of cores depends broadly on the logic underlying the argumentation system (i.e., $(\mathcal{L}, \mathrm{CN})$). We show that finiteness is ensured by logics in which any consistent finite set of formulae has finitely many logically non-equivalent consequences when the knowledge base is finite. Two examples of such logics are Parry's (1989) and the fragment of intuitionistic logic (introduced by McKinsey and Tarki) studied by McCall (1962).

Theorem 9. *Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ be an argumentation system built over a knowledge base Σ (i.e., $\mathcal{A} \subseteq \mathrm{Arg}(\Sigma)$). If $\mathrm{Cncs}(\Sigma)/\equiv$ is finite, then every core of $\mathcal{F}$ is finite.*

To sum up, under some reasonable conditions on the attack relation and the logic, any argumentation system has finite and equivalent cores.

Corollary 10. *Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ and $\mathcal{F}' = (\mathcal{A}', \mathcal{R}')$ be two argumentation systems s.t. $\mathcal{R}$ and $\mathcal{R}'$ satisfy $C1b$ and $C2b$ and* $\mathrm{Cncs}(\Sigma)/\equiv$ *is finite. If $\mathcal{F}'$ is a core of $\mathcal{F}$, then $\mathcal{F}'$ is finite and $\mathcal{F} \equiv_{EQ1b} \mathcal{F}'$.*

4.1 *Core(s) in propositional logic*

The previous section has shown that argumentation systems which are built under some particular logics have finite cores. Propositional logic is not one of them since the set $\mathrm{Cncs}(\Sigma)/\equiv$ is not finite. Let us consider the following counter-example.

Example 13. *Let $(\mathcal{L}, \mathrm{CN})$ be propositional logic and let $\Sigma = \{x\}$. The set* $\mathrm{Cncs}(\Sigma)$ *contains the following formulae: $x, x \vee z_1, x \vee z_2, x \vee z_3 \ldots$ It is thus infinite.*

Thus, under propositional logic, the set of all arguments that can be built from a finite knowledge base is infinite. The proof of the following property follows from the idea of the previous example.

Property 13. *Let $(\mathcal{L}, \mathrm{CN})$ be propositional logic and Σ a finite knowledge base having at least one consistent formula. The set* $\mathrm{Arg}(\Sigma)$ *is infinite.*

Despite the previous properties (on $\mathrm{Cncs}(\Sigma)$ and $\mathrm{Arg}(\Sigma)$), it is possible to define *finite cores* for any argumentation system under propositional logic. The idea is to understand the reasons of infiniteness and to try to avoid them. There are several sources of infiniteness of the set of arguments. The first one is the fact of *duplicating* several arguments with the same support and equivalent conclusions. For instance, the arguments $\langle\{x\}, x \vee y\rangle$, $\langle\{x\}, \neg x \to y\rangle$ and $\langle\{x\}, (\neg x \to y) \vee (x \vee \neg x)\rangle$ are built from $\Sigma = \{x, y\}$ and are in some sense redundant, or *equivalent* with respect to relation $\approx$. Similar remark holds for the two arguments $\langle\{x\}, x\rangle$ and $\langle\{x\}, x \wedge x\rangle$. It is easy to see that the number of such arguments is infinite. Property 8 shows that such arguments have the same status in an argumentation system whose attack relation verifies the two properties $C1b$ and $C2b$.

The second source of infiniteness of a set of arguments is due to atoms that have no occurrence within the knowledge base Σ but occur in conclusions of arguments. For instance, the two arguments $\langle\{x\}, x \vee z\rangle$ and $\langle\{x\}, x \vee z \vee w\rangle$ belong to the set $\mathrm{Arg}(\Sigma)$ although z and w do not occur in $\Sigma = \{x, y\}$. This section shows that such arguments have no impact on the other arguments of $\mathrm{Arg}(\Sigma)$.

Another source of infiniteness might be an infinite knowledge base Σ. It can contain an infinite amount of non redundant information and in such a case it is impossible to find a finite core of the corresponding argumentation system. That is why, throughout the paper, we suppose that Σ is finite.

In order to illustrate how to deal with the sources of infiniteness on a concrete example, the reminder of the section presents a detailed study of the case when a particular attack relation (called *assumption attack*) is used together with *stable semantics*. In the next section, we show how some of the results can be generalised to a large class of logics. Let us first introduce some notations.

Notations: $\mathrm{Atoms}(\Sigma)$ is the set of atoms occurring in Σ. $\mathrm{Arg}(\Sigma)_{\downarrow}$ is the subset of $\mathrm{Arg}(\Sigma)$ that contains only arguments with conclusions based on $\mathrm{Atoms}(\Sigma)$. For instance, for $\Sigma = \{x, y\}$, $\mathrm{Atoms}(\Sigma) = \{x, y\}$. Thus, an argument such as $(\{x\}, x \vee z \vee w)$ does not belong to the set $\mathrm{Arg}(\Sigma)_{\downarrow}$.

We now define the attack relation we use in this section.

Definition 15 (Assumption attack). *Let* Σ *be a propositional knowledge base and* $a, b \in \text{Arg}(\Sigma)$*. The argument* a undermines b*, denoted* $a\mathcal{R}_{as}b$*, iff* $\exists x \in \text{Supp}(b)$ *s.t.* $\text{Conc}(a) \equiv \neg x$*.*

It is worth noticing that this relation satisfies the two properties $C1b$ and $C2b$.

Property 14. *The relation* $\mathcal{R}_{as}$ *verifies the two properties* $C1b$ *and* $C2b$*.*

Now, note that the set $\text{Arg}(\Sigma)_{\downarrow}$ is infinite (due to equivalent arguments). In what follows, we show that its arguments have the same status in the two systems $\mathcal{F} = (\text{Arg}(\Sigma), \mathcal{R})$ and $\mathcal{F}_{\downarrow} = (\text{Arg}(\Sigma)_{\downarrow}, \mathcal{R}_{\downarrow})$ (where $\mathcal{R}_{\downarrow}$ is of course the restriction of $\mathcal{R}$ to $\text{Arg}(\Sigma)_{\downarrow}$). The first result shows that arguments that use external variables (i.e., variables which are not in $\text{Atoms}(\Sigma)$) in their conclusions can be omitted from the reasoning process.

Theorem 11. *Let* $\mathcal{F} = (\text{Arg}(\Sigma), \mathcal{R}_{as})$ *be an argumentation system built over a propositional knowledge base* Σ*, and* $\mathcal{F}_{\downarrow} = (\text{Arg}(\Sigma)_{\downarrow}, \mathcal{R}_{as\downarrow})$ *its sub-system. For all* $a \in \text{Arg}(\Sigma)_{\downarrow}$*,* $\text{Status}(a, \mathcal{F}) = \text{Status}(a, \mathcal{F}_{\downarrow})$ *under stable semantics.*

Moreover, we show next that their status is still known. It is that of any argument in $\text{Arg}(\Sigma)_{\downarrow}$ with the same support.

Theorem 12. *Let* $\mathcal{F} = (\text{Arg}(\Sigma), \mathcal{R}_{as})$ *be an argumentation system built over a propositional knowledge base* Σ*. For all* $a \in \text{Arg}(\Sigma) \setminus \text{Arg}(\Sigma)_{\downarrow}$*, under stable semantics,* $\text{Status}(a, \mathcal{F}) = \text{Status}(b, \mathcal{F})$ *where* $b \in \text{Arg}(\Sigma)_{\downarrow}$ *and* $\text{Supp}(a) \approx \text{Supp}(b)$*.*

To sum up, the two previous theorems clearly show that one can use the sub-system $\mathcal{F}_{\downarrow} = (\text{Arg}(\Sigma)_{\downarrow}, \mathcal{R}_{as\downarrow})$ instead of $\mathcal{F} = (\text{Arg}(\Sigma), \mathcal{R}_{as})$ without losing any information. This system is still infinite due to redundant arguments. However, we prove next that the set $\text{Arg}(\Sigma)_{\downarrow}$ is partitioned into a finite number of equivalence classes with respect to the equivalence relation $\approx$.

Theorem 13. *For every propositional knowledge base* Σ*, it holds that* $|\text{Arg}(\Sigma)_{\downarrow}/\approx| \leq 2^n \cdot 2^{2^m}$*, where* $n = |\Sigma|$ *and* $m = |\text{Atoms}(\Sigma)|$*.*

This result is of great importance since it shows how it is possible to partition an infinite set of arguments into a finite number of classes. Note that each class may contain an infinite number of arguments. An example of such an infinite class is the one which contains (but is not limited to) all the arguments having $\{x\}$ as a support and $x, x \wedge x, \ldots$ as conclusions. A consequence of this result is that the cores of an argumentation system which considers only the set $\text{Arg}(\Sigma)_{\downarrow}$ of arguments are finite.

Theorem 14. *Let* Σ *be a propositional knowledge base and* $\mathcal{F} = (\mathcal{A}, \mathcal{R}_{as})$ *be an argumentation system such that* $\mathcal{A} \subseteq \text{Arg}(\Sigma)_{\downarrow}$*. Then every core of* $\mathcal{F}$ *is finite.*

Since the attack relation $\mathcal{R}_{as}$ satisfies the two properties $C1b$ and $C2b$, then from Theorem 7, an argumentation system that does not accept external variables in its arguments is equivalent to any of its cores.

Corollary 15. *Let* Σ *be a propositional knowledge base and* $\mathcal{F} = (\mathcal{A}, \mathcal{R}_{as})$ *be an argumentation system such using stable semantics that* $\mathcal{A} \subseteq \text{Arg}(\Sigma)_{\downarrow}$*.* $\mathcal{F} \equiv_{EQ1b} \mathcal{F}'$ *where* $\mathcal{F}'$ *is a core of* $\mathcal{F}$*.*

Note that no core is equivalent to the original argumentation $\mathcal{F} = (\text{Arg}(\Sigma), \mathcal{R}_{as})$ with respect to $EQ1b$. This is because the set $\text{Output}(\mathcal{G})/\equiv$ of any core $\mathcal{G}$ is finite while $\text{Output}(\mathcal{F})/\equiv$ is infinite (due to conclusions containing atoms not occurring in Σ). However, the next result shows that it is possible to compute the output of the original argumentation system from the output of one of its cores.

Theorem 16. *Let $\mathcal{F}$ be an argumentation system built over a propositional knowledge base Σ using stable semantics and let $\mathcal{G}$ be one of its cores.*

$$\text{Output}_{sc}(\mathcal{F}) = \{x \in \mathcal{L} \ s.t. \ \text{Output}_{sc}(\mathcal{G}) \vdash x\}.$$

An important question now is how to choose a core? A simple solution would be to pick exactly one formula from each set of logically equivalent formulae. Since a lexicographic order on set $\mathcal{L}$ is usually available, we can take the first formula from that set according to that order. Instead of defining a lexicographic order, one could also choose to take the disjunctive (or conjunctive) normal form of a formula.

4.2 *On the finiteness of core in other logics*

The previous section presented a study of cores on a concrete example (propositional logic, assumption attack, stable semantics). In this section, we show that if the atoms not appearing in Σ are not used in the conclusions of arguments, there is a large class of logics having finite cores. The main technical challenge is that the notion of an abstract logic defined by Tarski is too abstract: namely there is no notion of atom or variable. So it is not possible to speak of omitting atoms appearing in Σ. Thus, we will base our result ona the notion of an *algebra*. First, recall that an algebra is a tuple $(A, (f_i)_{i \in I})$ where each f_i is an n_i-ary operation over A. The *similarity type* of the algrebra is $(n_i)_{i \in I}$.

In this section, we consider only logics satisfying the following four conditions, that we call bounded algebraic logics.

(1) The language of such a logic is a term algebra $\mathcal{L} = (F, o_1, \ldots, o_n)$ such that $n \in \omega$ and $n_i \in \omega$ for $i = 1..n$ (that is, the language only has finitely many logical symbols, none of them is infinite in character: there is no infinitary disjunction, no infinitary conjunction, ...).
(2) A model of such a logic can be characterized as an homomorphism from $\mathcal{L}$ to an algebra whose similarity type is exactly that of $\mathcal{L}$ (not all such homomorphisms need be models of the logic).
(3) Completeness holds (that is, $\text{CN}(\Phi) = \text{CN}(\Psi)$ iff $m(\Phi) = m(\Psi)$ in all models m of $(\mathcal{L}, \text{CN})$).
(4) Such a logic is to enjoy absorption laws for $o_1, \ldots, o_n$ as follows.

Absorption laws for unary logical symbols:
Let $\{o_1, \ldots, o_m\}$ be the set of unary operators of $\mathcal{L}$. We call a *prefix* is a finite sequence of operators. The logic is supposed to satisfy the following condition: For every subset $\{o_i, \ldots, o_k\}$ of the set of unary operators, for every atomic formula α, for every model m of $(\mathcal{L}, \text{CN})$, there exists $l < \omega$ such that for every prefix P over $\{o_i, \ldots, o_k\}$ there exists a prefix P' over $\{o_1, \ldots, o_m\}$ such that $length(P') \leq l$ and $m(P(\gamma)) = m(P'(\gamma))$.

The system of absorption laws which is required for such logics need not be non-redundant nor optimal in any way, all is required is existence (possibly through equivalence).

Since the number of unary operators is finite, there exists $K < \omega$ which is an upper bound of the length of prefixes P' for different subsets of unary operators.

As an illustration, take propositional intuitionistic logic. There is only one unary operator, namely $\neg$. We have $l = 2$ since $\gamma \equiv \gamma$, $\neg\gamma \equiv \neg\gamma$, $\neg\neg\gamma \equiv \neg\neg\gamma$, $\neg\neg\neg\gamma \equiv \neg\gamma$, $\neg\neg\neg\neg\gamma \equiv \neg\neg\gamma$, $\neg\neg\neg\neg\neg\gamma \equiv \neg\gamma$ etc.

As another example, in propositional modal logic $S5$, such an absorption law is $\diamond\neg\diamond\gamma \equiv \neg\diamond\gamma$. In this case, we have $l = 3$.

For a formula Φ, the sublanguage $\mathcal{L}_\Phi$ obtained by using only non-logical symbols occurring in Φ and the logical symbols from $\mathcal{L}$ is easily defined (as the subalgebra of $\mathcal{L}$ generated by the non-logical symbols occurring in Φ). Also, since a formula is a member of the term algebra $\mathcal{L}$, the notion of a subformula coincides with the notion of a subterm in $\mathcal{L}$.

Given a formula Θ from $\mathcal{L}$, let $x_1, \ldots, x_l$ be all the atoms occurring in it. Let F_k define the set of all formulas from $\mathcal{L}_\Theta$ in which each x_j occurs at most k times and no o_i occur that would be such that $n_i < 2$. Clearly, for every k, we have that F_k is finite. Let us define F_k^+ as the set obtained as follows: For a formula φ, replace every sub-formula θ in φ by $P\theta$ where P ranges over all prefixes of length less or equal to K. Do this for every formula φ in F_k.

Lemma 17. *For every k, F_k^+ is finite.*

Absorption laws for n-ary logical symbols:
It is required that the logic satisfies the following condition:
Given a formula Θ, there exists $k < \omega$ such that for every non unary operator o_i, for every $\gamma_1, \ldots, \gamma_n \in F_k^+$, there exists $\delta \in F_k^+$ such that for every model m of $(\mathcal{L}, \mathrm{CN})$ we have

$$m(o_i(\gamma_1, \ldots, \gamma_n)) = m(\delta).$$

As an illustration, in propositional classical logic, such an absorption law is $(\beta \wedge \gamma) \vee \gamma \equiv \gamma$.

Please observe that, in a number of logics, if $o_i(\gamma_1, \ldots, \gamma_{n_i})$ is in F_k^+ then it may happen that the corresponding absorption law be identity.

Theorem 18. *For every formula $\alpha \in \mathcal{L}_\Theta$, there exists $\sigma \in F_k^+$ s. t. $\mathrm{CN}(\alpha) = \mathrm{CN}(\sigma)$.*

We define $\mathrm{CN}_{\mathcal{L}_\Theta}(\Theta)$ to be $\mathrm{CN}(\Theta) \cap \mathcal{L}_\Theta$. By applying the previous theorem in view of the lemma we obtain the following result.

Corollary 19. *Given a formula Θ from $\mathcal{L}$, $\mathrm{CN}_{\mathcal{L}_\Theta}(\Theta)$ is partitioned into finitely many* CN-*equivalence classes.*

The main theorem of this section is now a direct consequence of the previous result.

Theorem 20. *Let $(\mathcal{L}, \mathrm{CN})$ be a bounded algebraic logic. Let Σ be a finite set of formulas from $\mathcal{L}$. Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ be an argumentation system such that $\mathcal{A}$ is a set of arguments whose conclusions are all in $\mathrm{CN}(\Sigma) \cap \mathcal{L}_\Sigma$. Then, every core of $\mathcal{F}$ is finite.*

Note that unlike Wojcicki (1988) that calls algebraic logics those systems which satisfy Tarki's finiteness axiom, we restrict the meaning of algebraic logic to those that admit an algebraic semantics in the above direct manner (excluding for instance semantics based on cylindrical algebras as were designed for predicate first-order logic). Algebraic semantics for well-known logics can be found in the literature (Rasiowa & Sikorski, 1963).

4.3 *Dynamics of argument status*

Several works studied the dynamics of an argumentation system. They mainly investigate how the acceptability or status of an argument may evolve when the argumentation system is extended by new arguments. For instance, Amgoud and Vesic (2012) show that an argument may be sceptically accepted in a system, and becomes rejected in an extended version of the system. Similarly, an argument may be rejected or credulously accepted in a system and becomes sceptically accepted in an extended system. It was shown that the same phenomenon occurs when arguments are removed from an argumentation system (Bisquert, Cayrol, Saint-Cyr, & Lagasquie, 2011). In what follows, we show that the notion of core is at the heart of this change in arguments' statuses.

Throughout this section, we consider a fixed Tarskian logic $(\mathcal{L}, \mathrm{CN})$ and an attack relation $\mathcal{R}(\mathcal{L}) \subseteq \mathrm{Arg}(\mathcal{L}) \times \mathrm{Arg}(\mathcal{L})$ which satisfies the two properties $C1b$ and $C2b$. Given an argumentation system $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ and a set $\mathcal{E}$ of arguments,

- $\mathcal{F} \oplus \mathcal{E} = (\mathcal{A}', \mathcal{R}')$ with $\mathcal{A}' = \mathcal{A} \cup \mathcal{E}$ and $\mathcal{R}' = \mathcal{R}(\mathcal{L})|_{\mathcal{A}'}$.
- $\mathcal{F} \ominus \mathcal{E} = (\mathcal{A}', \mathcal{R}')$ with $\mathcal{A}' = \mathcal{A} \setminus \mathcal{E}$ and $\mathcal{R}' = \mathcal{R}(\mathcal{L})|_{\mathcal{A}'}$.

Before presenting the formal results, let us introduce a new definition.

Definition 16. *Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ and $\mathcal{G}$ be argumentation systems. $\mathcal{F}$ contains a core of $\mathcal{G}$ iff there exists an argumentation system $\mathcal{H} = (\mathcal{A}_h, \mathcal{R}_h)$ s.t. $\mathcal{A}_h \subseteq \mathcal{A}$ and $\mathcal{R}_h \subseteq \mathcal{R}$ and $\mathcal{H}$ is a core of $\mathcal{G}$.*

The next result shows that if an argumentation system contains a core of its complete version, then adding new arguments does not impact on the status of existing arguments.

Theorem 21. *Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ be an argumentation system built over a knowledge base Σ such that $\mathcal{R}$ satisfies C1b and C2b. If $\mathcal{F}$ contains a core of $\mathcal{G} = (\mathrm{Arg}(\mathrm{Base}(\mathcal{A})), \mathcal{R}(\mathcal{L})|_{\mathrm{Arg}(\mathrm{Base}(\mathcal{A}))})$, then for all $\mathcal{E} \subseteq \mathrm{Arg}(\mathrm{Base}(\mathcal{A}))$,*

- $\mathcal{F} \equiv_{EQ1b} \mathcal{F} \oplus \mathcal{E}$
- $\forall a \in \mathcal{A}, \mathrm{Status}(a, \mathcal{F}) = \mathrm{Status}(a, \mathcal{F} \oplus \mathcal{E})$
- $\forall e \in \mathcal{E} \setminus \mathcal{A}, \mathrm{Status}(e, \mathcal{F} \oplus \mathcal{E}) = \mathrm{Status}(a, \mathcal{F})$, *where $a \in \mathcal{A}$ is any argument s.t.* $\mathrm{Supp}(a) \approx \mathrm{Supp}(e)$.

It is clear that the previous result holds when $\mathcal{F}$ is itself a core of $\mathcal{G}$. The following example shows that when a system does not contain a core of the system built over its base, new arguments may change the status of the existing ones.

Example 14. *Let $(\mathcal{L}, \mathrm{CN})$ be propositional logic and let the attack relation $\mathcal{R}(\mathcal{L})$ be the assumption attack relation. Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ be an argumentation system such that $\mathcal{A} = \{a_1 = (\{strad, strad \to exp\}, exp), a_2 = (\{\neg strad\}, \neg strad)\}$. Recall that $\mathcal{R} = \mathcal{R}(\mathcal{L})|_{\mathcal{A}}$. thus, $\mathcal{R} = \{(a_2, a_1)\}$. The argument a_2 is sceptically accepted whereas a_1 is rejected. Let $e = (\{strad\}, strad)$. It is clear that $e \in \mathrm{Arg}(\mathrm{Base}(\mathcal{A}))$. However, the status of each of a_1 and a_2 changes in the system $\mathcal{F} \oplus \{e\}$. Namely, both arguments become credulously accepted.*

The previous example illustrated a situation when an argumentation system $\mathcal{F}$ does not contain a core of the system constructed from its base. This means that not all available information is represented in $\mathcal{F}$; thus, it is not surprising that it is possible to revise arguments' statuses. In what follows, we provide also a situation where removing arguments from $\mathcal{F}$ will not impact the status of arguments in $\mathcal{F}$.

Theorem 22. *Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ be an argumentation system built over a knowledge base Σ such that $\mathcal{R}$ satisfies C1b and C2b and let $\mathcal{E} \subseteq \mathcal{A}$. If $\mathcal{F} \ominus \mathcal{E}$ contains a core of $\mathcal{G} = (\mathrm{Arg}(\mathrm{Base}(\mathcal{A})), \mathcal{R}(\mathcal{L})|_{\mathrm{Arg}(\mathrm{Base}(\mathcal{A}))})$, then:*

- $\mathcal{F} \equiv_{EQ1b} \mathcal{F} \ominus \mathcal{E}$
- $\forall a \in \mathcal{A} \setminus \mathcal{E}$, $\mathrm{Status}(a, \mathcal{F}) \approx \mathrm{Status}(a, \mathcal{F} \ominus \mathcal{E})$.

The obvious consequence of the above result is that if $\mathcal{F} \ominus \mathcal{E}$ is itself one of the cores of $\mathcal{G}$. The statuses of its arguments are not changed after removing arguments from $\mathcal{E}$.

5. Conclusion

In this paper, we tackled the question: *"When are two logic-based argumentation systems equivalent?"* We proposed various equivalence criteria. Some of them are shown to be syntax-dependent whereas others are more flexible and take advantage of equivalences between arguments and between formulae. The links between the criteria are largely investigated. Some of the results hold for any acceptability semantics and any attack relation, while others make reasonable assumptions on the attack relations or are shown under particular semantics. The comparative study revealed that there is one particular criterion which is both flexible ans general. Thus, in the second part of the paper, we only focused on this criterion. We studied under which conditions two systems are equivalent with respect to this criterion. We have shown how to pass from infinite argumentation systems to finite ones and how to replace a system by a proper sub-system without losing information.

It is worth mentioning that equivalence between arguments and sets of arguments was also studied from a computational complexity perspective (Wooldridge, Dunne, & Parsons, 2006). The authors focused on one particular argumentation system: the one that is built on propositional logic and that uses the assumption attack relation. According to this work, two arguments are logically equivalent if and only if their conclusions are logically equivalent. Thus, the two arguments $a = (\{y, y \to x\}, x)$ and $a' = (\{z, z \to x\}, x)$ are equivalent. Note that in our paper, those two arguments are not equivalent. We consider them not equivalent since they are based on different hypotheses. It can be the case that one of those hypotheses is attacked and not the other ones. For example, the argument $b = (\{\neg y\}, \neg y)$ attacks a but not a'. This example shows that the equivalence relation considered by Wooldridge et al. is too simplistic and is not sufficient to guarantee that all information from a knowledge base is represented in an argumentation system. Those authors also propose an equivalence criterion between sets of arguments. According to this criterion, two sets X and Y of arguments are equivalent if there is a bijection between them, i.e., a function f s.t. $\forall x \in X$, $f(x)$ is equivalent with x (using their equivalence criterion between arguments). In this paper, we proposed a more flexible criterion. Let us consider the following example: let $X = \{(\{x\}, x), (\{x\}, \neg\neg x)\}$ and $Y = \{(\{x\}, x)\}$. According to our criterion, the two sets X and Y are equivalent while they are not equivalent with respect to the criterion used by Wooldridge et al. (2006). Note that our criterion allows us to reduce an infinite system to a finite one, which is impossible if using the definition demanding for a bijection between the two sets.

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Appendix

Property 1 *Each criterion $\approx_i$ is an equivalence relation (with $i \in \{1, 2, 3\}$).*

Proof. The three relations are reflexive since relations $\cong$, $\equiv$ and $=$ are reflexive, they are symmetric since $\cong$, $\equiv$ and $=$ are symmetric and they are transitive since $\cong$, $\equiv$ and $=$ are transitive. □

Property 2 *Let $a, a' \in \text{Arg}(\mathcal{L})$.*

- *If $a \approx_1 a'$, then $a \approx_3 a'$*
- *If $a \approx_2 a'$, then $a \approx_3 a'$*

Proof. The claim follows from the fact that for every two sets of formulae X and Y, it holds that $X = Y$ implies $X \cong Y$ and $X \equiv Y$. □

Property 3 *For two argumentation systems $\mathcal{F}$ and $\mathcal{F}'$, if $\mathcal{F} \equiv_{EQi} \mathcal{F}'$ then $\mathcal{F} \equiv_{EQib} \mathcal{F}'$ with $i \in \{1, \ldots, 6\}$.*

Proof. The property follows from the observations that for every pair of sets of arguments $\mathcal{E}$, $\mathcal{E}'$, we know that $\mathcal{E} = \mathcal{E}'$ implies $\mathcal{E} \sim \mathcal{E}'$ and that for every pair of sets of formulae X, Y, it holds that $X = Y$ implies $X \cong Y$. □

Property 4 *For all $i \in \{1, \dots, 6\}$, the criterion EQi (resp. EQib) is an equivalence relation.*

Proof. The result follows from the elementary properties of bijections, together with the fact that both $\sim$ and $\cong$ are equivalence relations. □

Property 5 *Let $\mathcal{R}$ be an attack relation.*

- *If $\mathcal{R}$ satisfies $C1b$ then it satisfies $C1$.*
- *If $\mathcal{R}$ satisfies $C2b$ then it satisfies $C2$.*

Proof. The proof follows directly from the two following observations: first, for two formulae φ and ψ, it holds that $\varphi = \psi$ implies $\varphi \equiv \psi$; second, for two sets of formulae X and Y, if $X = Y$, then $X \cong Y$. □

Property 6 *Let $(\mathcal{A}, \mathcal{R})$ be an argumentation system s.t. $\mathcal{R}$ enjoys $C1b$ and $C2b$. For all $a, a', b, b' \in \mathcal{A}$, ($a \approx a'$ and $b \approx b'$) implies ($a\mathcal{R}b$ iff $a'\mathcal{R}b'$).*

Proof. Let $a \approx a'$ and $b \approx b'$ and let $a\mathcal{R}b$. Since $\text{Supp}(b) \cong \text{Supp}(b')$ then from C2b we have that $a\mathcal{R}b'$. From C1b and $\text{Conc}(a) \equiv \text{Conc}(a')$, we obtain $a'\mathcal{R}b'$. To show that $a'\mathcal{R}b'$ implies $a\mathcal{R}b$ is similar. □

Property 7 *Let $(\mathcal{A}, \mathcal{R})$ be an argumentation system s.t. $\mathcal{R}$ enjoys $C2b$. For all $a, a' \in \mathcal{A}$, if $a \approx a'$, then $\forall \mathcal{E} \in \text{Ext}(\mathcal{F})$, $a \in \mathcal{E}$ iff $a' \in \mathcal{E}$.*

Proof. To prove this result, we use the notion of complete labelling (Caminada, 2006a). Since arguments a and a' have the same sets of attackers, then for every complete labelling L we have $L(a) = L(a')$. This means that for every complete extension $\mathcal{E}$, it holds that $a \in \mathcal{E}$ if and only $a' \in \mathcal{E}$. The proof follows from the fact that stable, preferred and grounded extensions are also complete extensions. □

Property 8 *Let $(\mathcal{A}, \mathcal{R})$ be an argumentation system s.t. $\mathcal{R}$ enjoys $C1b$ and $C2b$. For all $a, a' \in \mathcal{A}$, if $a \approx a'$, then* $\text{Status}(a, \mathcal{F}) = \text{Status}(a', \mathcal{F})$.

Proof. Let $x \in \{c, p, s, g\}$ and denote by $\text{Ext}_x(\mathcal{F})$ the set of extensions under this semantics. From Property 7, we see that: for all $\mathcal{E}_i \in \text{Ext}_x(\mathcal{F})$ it holds that $a \in \mathcal{E}_i$ if and only if $a' \in \mathcal{E}_i$. Consequently,

- for all $\mathcal{E}_i \in \text{Ext}_x(\mathcal{F})$, $a \in \mathcal{E}_i$ if and only if for all $\mathcal{E}_i \in \text{Ext}_x(\mathcal{F})$, $a' \in \mathcal{E}_i$
- for all $\mathcal{E}_i \in \text{Ext}_x(\mathcal{F})$, $a \notin \mathcal{E}_i$ if and only if for all $\mathcal{E}_i \in \text{Ext}_x(\mathcal{F})$, $a' \notin \mathcal{E}_i$

The proof now follows directly from those two observations. □

Property 9 *Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$, $\mathcal{F}' = (\mathcal{A}', \mathcal{R}')$ be two argumentation systems built from the same logic $(\mathcal{L}, \text{CN})$ such that $\mathcal{R}$ and $\mathcal{R}'$ enjoy $C1b$ and $C2b$. If $\mathcal{F} \equiv_{EQ1b} \mathcal{F}'$, then for all $a \in \mathcal{A}$ and for all $a' \in \mathcal{A}'$, if $a \approx a'$ then* $\text{Status}(a, \mathcal{F}) = \text{Status}(a', \mathcal{F}')$.

Proof. Let $x \in \{c, p, s, g\}$. If $\mathcal{F}$ has no extensions under semantics x, then all the arguments of $\mathcal{F}$ and $\mathcal{F}'$ are rejected. In the rest of the proof we study the case $\text{Ext}_x(\mathcal{F}) \neq \emptyset$. Let f be the bijection from EQ1b and let us prove that for every $\mathcal{E} \in \text{Ext}_x(\mathcal{F})$, $a \in \mathcal{E}$ if and only if $a' \in f(\mathcal{E})$.

Let $\mathcal{E} \in \text{Ext}_x(\mathcal{F})$ and let $a \in \mathcal{E}$. From EQ1b, we conclude that there exists $a'' \in f(\mathcal{E})$ such that $a \approx a''$. Since $\approx$ is transitive, then $a' \approx a''$. Thus, from Property 7, we have that $a' \in f(\mathcal{E})$.

Let us now suppose that $a' \in f(\mathcal{E})$ and prove that $a \in \mathcal{E}$. From EQ1b, there exists $a''' \in \mathcal{E}$ such that $a' \approx a'''$. From the transitivity of $\approx$, $a''' \approx a$. From Property 7, $a \in \mathcal{E}$.

Thus, we see that for every extension $\mathcal{E}$ of $\mathcal{F}$, we have that $a \in \mathcal{E}$ if and only if $a' \in f(\mathcal{E})$. From this, we can conclude that:

- $a \in \bigcap_{\mathcal{E}_i \in \text{Ext}_x(\mathcal{F})} \mathcal{E}_i$ iff $a' \in \bigcap_{\mathcal{E}'_i \in \text{Ext}_x(\mathcal{F}')} \mathcal{E}'_i$
- $a \in \bigcup_{\mathcal{E}_i \in \text{Ext}_x(\mathcal{F})} \mathcal{E}_i$ iff $a' \in \bigcup_{\mathcal{E}'_i \in \text{Ext}_x(\mathcal{F}')} \mathcal{E}'_i$
- $a \notin \bigcup_{\mathcal{E}_i \in \text{Ext}_x(\mathcal{F})} \mathcal{E}_i$ iff $a' \notin \bigcup_{\mathcal{E}'_i \in \text{Ext}_x(\mathcal{F}')} \mathcal{E}'_i$

In other words, if a is sceptically accepted, a' is sceptically accepted, if a is credulously accepted, a' is credulously accepted and if a is rejected then a' is rejected. This ends the proof. □

Property 10 *Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ and $\mathcal{F}' = (\mathcal{A}', \mathcal{R}')$ be two argumentation systems. If $\mathcal{F}'$ is a core of $\mathcal{F}$, then $\mathcal{A} \sim \mathcal{A}'$.*

Proof. The property follows directly from Definition 14. □

Property 11 *Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ be an argumentation system and $\mathcal{F}' = (\mathcal{A}', \mathcal{R}')$ its core. If $\mathcal{R}$ satisfies $C1b$ and $C2b$ then:*

- *If $a \in \mathcal{A}'$, then* $\text{Status}(a, \mathcal{F}) = \text{Status}(a, \mathcal{F}')$,
- *If $a \notin \mathcal{A}'$, then* $\text{Status}(a, \mathcal{F}) = \text{Status}(b, \mathcal{F}')$ *for some $b \in \mathcal{A}'$ with $a \approx b$.*

Proof.

- From Theorem 7, $\mathcal{F} \equiv_{EQ1b} \mathcal{F}'$. From Property 9, $\text{Status}(a, \mathcal{F}) = \text{Status}(a, \mathcal{F}')$.
- From the first part of the property, $\text{Status}(b, \mathcal{F}) = \text{Status}(b, \mathcal{F}')$. Let us show that $\text{Status}(a, \mathcal{F}) = \text{Status}(b, \mathcal{F})$. Since $a \approx b$ and $\mathcal{R}$ satisfies $C1b$ and $C2b$, then a and b are attacked by the same arguments. This means that for every complete labelling (Caminada, 2006a) L, it holds that $L(a) = L(b)$. Since stable, preferred and grounded extensions are complete extensions, then $\text{Status}(a, \mathcal{F}) = \text{Status}(b, \mathcal{F})$ with respect to any of those semantics. Thus, $\text{Status}(a, \mathcal{F}) = \text{Status}(b, \mathcal{F}')$.

□

Property 13 *Let $(\mathcal{L}, \text{CN})$ be propositional logic and Σ a finite knowledge base having at least one consistent formula. The set $\text{Arg}(\Sigma)$ is infinite.*

Proof. Let $\varphi \in \Sigma$ be a consistent formula and let without loss of generality ψ_1, ψ_2, ... be the atoms not appearing in φ. Set $\text{Arg}(\Sigma)$ contains all the following arguments: $(\{\varphi\}, \varphi \vee \psi_1)$, $(\{\varphi\}, \varphi \vee \psi_2)$, $(\{\varphi\}, \varphi \vee \psi_3)$, ... Thus, $\text{Arg}(\Sigma)$ is infinite. □

Lemma 17 *For every k, F_k^+ is finite.*

Proof. Clearly, each formula from F_k offers finitely many occurrences to be replaced and there are finitely many substituting strings. Therefore, each formula from F_k gives rise to finitely many formulas in F_k^+. Since F_k is finite, it then follows that so is F_k^+. □

Lemma 23. *Let $(\mathcal{A}_c, \mathcal{R}_c)$ be a core of $\mathcal{F}_\downarrow = (\mathcal{A}_\downarrow = \text{Arg}(\Sigma)_\downarrow, \mathcal{R}_\downarrow = \mathcal{R}(\mathcal{L})|_{\mathcal{A}_\downarrow})$ and let $\mathcal{A}_1$ be an arbitrary set which contains $\mathcal{A}_c$, i.e. $\mathcal{A}_c \subseteq \mathcal{A}_1 \subseteq \text{Arg}(\Sigma)$. We define $\mathcal{R}_1 = \mathcal{R}|_{\mathcal{A}_1}$, as expected, and $\mathcal{F}_1 = (\mathcal{A}_1, \mathcal{R}_1)$. Let $S_1, \ldots, S_n$ be all the maximal consistent subsets of Σ, and let $\mathcal{E}_1 = \text{Arg}(S_1) \cap \mathcal{A}_1, \ldots, \mathcal{E}_n = \text{Arg}(S_n) \cap \mathcal{A}_1$. Then, $\text{Ext}(\mathcal{F}_1) = \{\mathcal{E}_1, \ldots, \mathcal{E}_n\}$.*

Proof. We will first prove that for any maximal consistent subset S_i of Σ, the set $\mathcal{E}_i = \text{Arg}(S_i) \cap \mathcal{A}_1$ is a stable extension of $\mathcal{F}_1$. It is easy to see that if S_i is consistent then $\text{Arg}(S_i)$ is conflict-free. Let us prove that $\mathcal{E}_i$ attacks any argument in $\mathcal{A}_1 \setminus \mathcal{E}_i$. Let $a' \in \mathcal{A}_1 \setminus \mathcal{E}_i$. Since $a' \notin \mathcal{E}_i$, then $\exists h \in \text{Supp}(a')$ s.t. $h \notin S_i$. Since $\text{Supp}(a') \subseteq \Sigma$ and S_i is a maximal consistent subset of Σ, it follows that $S_i \cup \{h\}$ is inconsistent. Then, there exists a minimal set $C \subseteq S_i$ s.t. $C \cup \{h\}$ is inconsistent. Let $a = (C, \neg h)$. Then, since a

uses only atoms from Σ (since $h \in \Sigma$) and since $(\mathcal{A}_c, \mathcal{R}_c)$ is a core of $\mathcal{F}_\downarrow$ then $\exists a_1 \in \mathcal{A}_c$ s.t. $a_1 \approx_1 a$. Since $\text{Supp}(a_1) \subseteq S_i$ then $a_1 \in \mathcal{E}_i$. Also, $a_1 \mathcal{R}_1 a'$. Hence, $\mathcal{E}_i$ is a stable extension of $\mathcal{F}_1$.

We will now prove that for any $\mathcal{E}' \in \text{Ext}(\mathcal{F}_1)$, there exists a maximal consistent subset of Σ, denoted S', s.t. $\mathcal{E}' = \text{Arg}(S') \cap \mathcal{A}_1$. To show this, we will show that: 1) $\text{Base}(\mathcal{E}')$ is consistent, 2) $\text{Base}(\mathcal{E}')$ is a maximal consistent set in Σ, 3) $\mathcal{E}' = \text{Arg}(\text{Base}(\mathcal{E}')) \cap \mathcal{A}_1$.

(1) Let $S' = \text{Base}(\mathcal{E}')$. Suppose that S' is an inconsistent set and let $C \subseteq S'$ be a minimal inconsistent subset of S'. Let $C = \{f_1, \ldots, f_k\}$, and let us construct the following argument: $a = (C \setminus \{f_1\}, \neg f_1)$. Since $\mathcal{E}'$ is conflict-free, then $a \notin \mathcal{E}'$ and $\nexists a_1 \in \mathcal{E}'$ s.t. $a_1 \approx_1 a$. Since $\mathcal{A}_c \subseteq A_1$, then there exists an argument $a_1 \in \mathcal{A}_1$ s.t. $a_1 \approx_1 a$. This means that, $a_1 \in \mathcal{A}_1 \setminus \mathcal{E}'$. Since $\mathcal{E}'$ is a stable extension, $\mathcal{E}'$ must attack a_1. Formally, $\exists a' \in \mathcal{E}'$ s.t. $a' \mathcal{R}_1 a_1$. So, $\text{Conc}(a') \equiv \neg f_2$ or $\text{Conc}(a') \equiv \neg f_3$, …, or $\text{Conc}(a') \equiv \neg f_k$. Without loss of generality, let $\text{Conc}(a') \equiv \neg f_k$. Since $f_k \in S'$, then there exists at least one argument a_k in $\mathcal{E}'$ s.t. $f_k \in \text{Supp}(a_k)$. Consequently, $\mathcal{E}'$ is not conflict-free, since a' attacks at least one argument in $\mathcal{E}'$. Contradiction. Hence, it must be that S' is consistent.

(2) Let $S' = \text{Base}(\mathcal{E}')$ and suppose that S' is not a maximal consistent set in Σ. According to (1) S' is consistent, hence $\exists f \in \Sigma \setminus S'$ s.t. $S' \cup \{f\}$ is consistent. Thus, for the argument $b = (\{f\}, f)$, we have that $\exists b_1 \in \mathcal{A}_1 \setminus \mathcal{E}'$ s.t. $b_1 \approx b$, but no argument in $\mathcal{E}'$ attacks b_1. (This is since $\neg f$ cannot be inferred from S'; consequently, no argument can be constructed from S' having its conclusion logically equivalent to $\neg f$.) Contradiction. Hence it must be that S' is a maximal consistent set.

(3) It is easy to see that for any set of arguments $\mathcal{E}'$, we have $\mathcal{E}' \subseteq \text{Arg}(\text{Base}(\mathcal{E}'))$. Since $\mathcal{S}' = \text{Base}(\mathcal{E}')$ is a consistent set, then set of arguments $\text{Arg}(\text{Base}(\mathcal{E}')) \cap \mathcal{A}_1$ must be conflict-free. From the fact that $\mathcal{E}'$ is a stable extension of $\mathcal{F}_1$, we conclude that the case $\mathcal{E}' \subsetneq \text{Arg}(\text{Base}(\mathcal{E}')) \cap \mathcal{A}_1$ is not possible (since every stable extension is a *maximal* conflict-free set).

We will now show that if S, S' are two different maximal consistent subsets of Σ, $\mathcal{E} = \text{Arg}(S) \cap \mathcal{A}_1$ and $\mathcal{E}' = \text{Arg}(S') \cap \mathcal{A}_1$, then $\mathcal{E} \neq \mathcal{E}'$. Without loss of generality, let $f \in S \setminus S'$. Let $a_f \in \mathcal{A}_1$ be an argument s.t. $\text{Supp}(a_f) = \{f\}$ and $\text{Conc}(a_f) \equiv f$. Such an argument must exist since $\mathcal{A}_1$ contains $\mathcal{A}_c$, and $(\mathcal{A}_c, R_c)$ is a core of $\mathcal{F}_\downarrow$. It is clear that $a \in \mathcal{E} \setminus \mathcal{E}'$, which shows that $\mathcal{E} \neq \mathcal{E}'$. This ends the proof. □

Theorem 1 *Let $\mathcal{F}$ and $\mathcal{F}'$ be two argumentation systems built on the same logic $(\mathcal{L}, \text{CN})$. Table 1 summarises the dependencies $(\mathcal{F} \equiv_x \mathcal{F}') \Rightarrow (\mathcal{F} \equiv_{x'} \mathcal{F}')$ under any of the reviewed semantics.*

Proof. Throughout the proof, we use notation $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ and $\mathcal{F}' = (\mathcal{A}', \mathcal{R}')$. We suppose any of the semantics from Definition 6.

First, note that EQ1 implies all the other criteria.

Let us now show that EQ1b implies EQ3b. Let $a \in \text{Cr}(\mathcal{F})$. Let us prove that $\exists a' \in \text{Cr}(\mathcal{F}')$ s.t. $a \approx a'$. Since $a \in \text{Cr}(\mathcal{F})$ then $\exists \mathcal{E} \in \text{Ext}(\mathcal{F})$ s.t. $a \in \mathcal{E}$. Let f be a bijection from EQ1b and let $\mathcal{E}' = f(\mathcal{E})$. From EQ1b, $\mathcal{E} \sim \mathcal{E}'$, thus $\exists a' \in \mathcal{E}'$ s.t. $a \approx a'$. This means that $\forall a \in \text{Cr}(\mathcal{F})$, $\exists a' \in \text{Cr}(\mathcal{F}')$ such that $a \approx a'$. To prove that $\forall a' \in \text{Cr}(\mathcal{F}')$, $\exists a \in \text{Cr}(\mathcal{F})$ such that $a \approx a'$ is similar. Thus, $\text{Cr}(\mathcal{F}) \sim \text{Cr}(\mathcal{F}')$.

Let us now show that EQ3b implies EQ5b. Let $\text{Cr}(\mathcal{F}) \sim \text{Cr}(\mathcal{F}')$ and let $\varphi \in \text{Output}_{cr}(\mathcal{F})$. Thus, there exists $a \in \text{Cr}(\mathcal{F})$ such that $\varphi = \text{Conc}(a)$. From $\text{Cr}(\mathcal{F}) \sim \text{Cr}(\mathcal{F}')$, we conclude that there exists $a' \in \text{Cr}(\mathcal{F}')$ such that $a \approx a'$. Thus, there exists $\varphi' \in \text{Output}_{cr}(\mathcal{F})$ such that $\varphi \equiv \varphi'$. Another direction of the implication is symmetric. Thus, we conclude that EQ3b implies EQ5b.

Since EQ1b implies EQ3b and EQ3b implies EQ5b, then EQ1b implies EQ5b.

Let us prove that EQ1b implies EQ6b. Suppose that EQ1b hold and let f be the bijection from this criterion. Let $\mathcal{S} \in \text{Bases}(\mathcal{F})$ and let $\mathcal{E} \in \text{Ext}(\mathcal{F})$ be an extension such that $\mathcal{S} = \text{Base}(\mathcal{E})$. Denote $\mathcal{E}' = f(\mathcal{E})$ and $S' = \text{Base}(\mathcal{E}')$. Since $\mathcal{E} \sim \mathcal{E}'$ then $S \cong S'$. Thus, EQ1b implies EQ6b.

From Property 3, we see that EQ2 implies EQ2b.

Let us show that EQ2 implies EQ4. Since $\text{Output}_{sc}(\mathcal{F}) = \{\text{Conc}(a) \mid a \in \text{Sc}(\mathcal{F})\}$ then we conclude that $\text{Sc}(\mathcal{F}) = \text{Sc}(\mathcal{F}')$ implies $\text{Output}_{sc}(\mathcal{F}) = \text{Output}_{sc}(\mathcal{F}')$. In other words, EQ2 implies EQ4.

Since EQ2 implies EQ4 and EQ4 implies EQ4b (Property 3), then EQ2 implies EQ4b.

Let us prove that EQ2 implies EQ4b. Let $\varphi \in \text{Output}_{sc}(\mathcal{F})$. Thus, there exists $a \in \text{Sc}(\mathcal{F})$, such that $\text{Conc}(a) = \varphi$. From EQ2b it follows that there exists $\varphi' \in \text{Output}_{sc}(\mathcal{F}')$ such that $\varphi \equiv \varphi'$. Consequently, there exists $a' \in \text{Sc}(\mathcal{F}')$ such that $\text{Conc}(a') = \varphi'$. This means that $\text{Output}_{sc}(\mathcal{F}) \cong \text{Output}_{sc}(\mathcal{F}')$. Hence EQ2b implies EQ4b.

From Property 3, EQ3 implies EQ3b.

Let us show that EQ3 implies EQ5. Since $\text{Output}_{cr}(\mathcal{F}) = \{\text{Conc}(a) \mid a \in \text{Cr}(\mathcal{F})\}$ then $\text{Cr}(\mathcal{F}) = \text{Cr}(\mathcal{F}')$ implies $\text{Output}_{cr}(\mathcal{F}) = \text{Output}_{cr}(\mathcal{F}')$. Hence EQ3 implies EQ5.

Since EQ3 implies EQ5 and EQ5 implies EQ5b (Property 3), then EQ3 implies EQ5b.

Note that we have already seen that EQ3b implies EQ5b.

That EQ4 implies EQ4b, EQ5 implies EQ5b and EQ6 implies EQ6b is shown by Property 3. This ends the proof. □

Theorem 2 *The links between the twelve equivalence criteria under grounded semantics are summarized in Table 2.*

Proof. Note that we only need to prove the links that do not exist in Theorem 1. Also, note that there is always exactly one extension, thus EQ1 coincides with EQ2 and EQ3. For the same reason, EQ1b coincides with EQ2b and EQ3b. EQ1b implies EQ2b since there is exactly one extension. Since EQ2b implies EQ4b in the general case, then EQ1b also implies EQ4b. Since EQ2 coincides with EQ1 and EQ1 implies all the other criteria, then EQ2 also implies all the other criteria. As already mentioned EQ2b is equivalent with EQ1b. The same holds for EQ2b and EQ3b. It is also easy to see that EQ2b implies EQ5b and EQ6b (since there is exactly one extension). EQ4 coincide with EQ5 for the above mentioned reason (that there is exactly one extension). The same applies to EQ4b and EQ5b. This ends the proof. □

Theorem 3 *Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$, $\mathcal{F}' = (\mathcal{A}', \mathcal{R}')$ be two argumentation systems built from the same logic $(\mathcal{L}, \text{CN})$, $\mathcal{R}$ and $\mathcal{R}'$ enjoy C1b and C2b. If $\mathcal{F} \equiv_{EQ1b} \mathcal{F}'$, then $\mathcal{F} \equiv_x \mathcal{F}'$*

with $x \in \{EQ2b, EQ4b\}$.

Proof. Suppose that the two systems are equivalent with respect to EQ1b and let us prove that EQ2b is satisfied. If $\text{Ext}(\mathcal{F}) = \emptyset$, then from EQ1b, $\text{Ext}(\mathcal{F}') = \emptyset$. In this case, EQ2b trivially holds, since $\text{Sc}(\mathcal{F}) = \text{Sc}(\mathcal{F}') = \emptyset$. Else, let $\text{Ext}(\mathcal{F}) \neq \emptyset$.

Let $\text{Sc}(\mathcal{F}) = \emptyset$ and let us prove that $\text{Sc}(\mathcal{F}') = \emptyset$. By means of contradiction, suppose the contrary and let $a' \in \text{Sc}(\mathcal{F}')$. Let $\mathcal{E}' \in \text{Ext}(\mathcal{F}')$. Argument a' is sceptically accepted, thus $a' \in \mathcal{E}'$. Let f be a bijection from EQ1b and let us denote $\mathcal{E} = f^{-1}(\mathcal{E}')$. From $\mathcal{F} \equiv_{EQ1b} \mathcal{F}'$, we obtain $\mathcal{E} \in \text{Ext}(\mathcal{F})$. Furthermore, $\mathcal{E} \sim \mathcal{E}'$, and, consequently, there exists $a \in \mathcal{E}$ s.t. $a \approx a'$. Property 9 implies that a is sceptically accepted in $\mathcal{F}$, contradiction.

Let $\text{Sc}(\mathcal{F}) \neq \emptyset$ and let us prove that $\text{Sc}(\mathcal{F}) \sim \text{Sc}(\mathcal{F}')$. Let $a \in \text{Sc}(\mathcal{F})$. Since $\mathcal{F} \equiv_{EQ1b} \mathcal{F}'$ and since a is in at least one extension, then there exists $a' \in \mathcal{A}'$ s.t. $a' \approx a$. Furthermore, Property 9 implies that a' is sceptically accepted in $\mathcal{F}'$. Thus for all $a \in \text{Sc}(\mathcal{F})$ there exists $a' \in \text{Sc}(\mathcal{F}')$ such that $a' \approx a$. To prove that for all $a' \in \text{Sc}(\mathcal{F}')$, there exists $a \in \text{Sc}(\mathcal{F})$ such that $a \approx a'$ is similar.
Since EQ2b implies EQ4b in the general case, as shown in Theorem 1, then we conclude that $\mathcal{F}$ and $\mathcal{F}'$ must also be equivalent with respect to EQ4b. □

Theorem 4 *Let* $(\mathcal{A}, \mathcal{R})$ *be an argumentation system built over* Σ*. If* Σ *is finite and* $\mathcal{R}$ *satisfies* $C2$*, then* $(\mathcal{A}, \mathcal{R})$ *has a finite number of extensions under all reviewed semantics.*

Proof. Let $x \in \{c, p, s, g\}$ and let $S_1, \ldots, S_n \subseteq \Sigma$ be all the consistent subsets of Σ. We will use the notation $\mathcal{A}_i = \{a \in \mathcal{A} \mid \text{Supp}(a) = S_i\}$, with $i \in \{1, \ldots, n\}$. (Note that some of the sets in $\mathcal{A}_1, \ldots, \mathcal{A}_n$ may be empty, but that is not important for the proof.) Let us prove that for every $\mathcal{E} \in \text{Ext}_x(\mathcal{F})$, for every two arguments $a, a' \in \mathcal{A}_i$ and a', we have $a \in \mathcal{E}$ if and only if $a' \in \mathcal{E}$. To prove this result, we rely on the notion of the complete labelling (Caminada, 2006a). Since a and a' are attacked by the same arguments, they have the same labels. Thus for every complete extension $\mathcal{E} \in \text{Ext}_c(\mathcal{F})$, we have $a \in \mathcal{E}$ if and only if $a' \in \mathcal{E}$. Since every stable, preferred and grounded extension is a complete one, then we conclude that for every $\mathcal{E} \in \text{Ext}_x(\mathcal{F})$ we have that $a \in \mathcal{E}$ if and only if $a' \in \mathcal{E}$. This means that for every $i \in \{1, \ldots, n\}$, for every extension $\mathcal{E} \in \text{Ext}_x(\mathcal{F})$, we have that $\mathcal{E}$ either contains all elements of $\mathcal{A}_i$ or neither of them. Formally, $\forall \mathcal{E} \in \text{Ext}(\mathcal{F})$, $\forall i \in \{1, \ldots, n\}$, we have $\mathcal{E} \cap \mathcal{A}_i = \mathcal{A}_i$ or $\mathcal{E} \cap \mathcal{A}_i = \emptyset$. Consequently, there is at most 2^n different extensions. □

Theorem 5 *Let* $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ *and* $\mathcal{F}' = (\mathcal{A}', \mathcal{R}')$ *be two argumentation systems s.t.* $\mathcal{A}, \mathcal{A}' \subseteq \text{Arg}(\mathcal{L})$ *and* $\mathcal{R} = \mathcal{R}_{\mathcal{L}}|_{\mathcal{A}}, \mathcal{R}' = \mathcal{R}_{\mathcal{L}}|_{\mathcal{A}'}$*. If* $\mathcal{R}_{\mathcal{L}}$ *satisfies* $C1b$ *and* $C2b$ *and* $\mathcal{A} \sim \mathcal{A}'$*, then* $\mathcal{F} \equiv_{EQ1b} \mathcal{F}'$*.*

Proof. Let $x \in \{c, p, s, g\}$. Define the function $f' : 2^{\mathcal{A}} \to 2^{\mathcal{A}'}$ as follows: $f'(B) = \{a' \in \mathcal{A}' \mid \exists a \in B \text{ s.t. } a' \approx a\}$. Let f be the restriction of f' to $\text{Ext}_x(\mathcal{F})$. We see that the image of this function is $\text{Ext}_x(\mathcal{F}')$ and that f is a bijection between $\text{Ext}_x(\mathcal{F})$ and $\text{Ext}_x(\mathcal{F}')$ satisfying EQ1b. □

Theorem 7 *Let* $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ *and* $\mathcal{F}' = (\mathcal{A}', \mathcal{R}')$ *be two argumentation systems s.t.* $\mathcal{R}$ *and* $\mathcal{R}'$ *satisfy* $C1b$ *and* $C2b$*. If* $\mathcal{F}'$ *is a core of* $\mathcal{F}$*, then* $\mathcal{F} \equiv_{EQ1b} \mathcal{F}'$*.*

Proof. The result is obtained by applying Theorem 5 on $\mathcal{F}$ and $\mathcal{F}'$. □

Theorem 9 *Let* $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ *be an argumentation system built over a knowledge base* Σ *(i.e.,* $\mathcal{A} \subseteq \text{Arg}(\Sigma)$*). If* $\text{Cncs}(\Sigma)/\equiv$ *is finite, then every core of* $\mathcal{F}$ *is finite.*

Proof. Let $\mathcal{F}' = (\mathcal{A}', \mathcal{R}')$ be a core of $\mathcal{F}$ and let us prove that $\mathcal{F}'$ is finite. Since Σ is finite, then $\{\text{Supp}(a) \mid a \in \mathcal{A}'\}$ must be finite. If for all $H \in \{\text{Supp}(a) \mid a \in \mathcal{A}'\}$, the set $\{a \in \mathcal{A}' \mid \text{Supp}(a) = H\}$ is finite, then the set $\mathcal{A}'$ is clearly finite. Else, there exists $H_0 \in \{\text{Supp}(a) \mid a \in \mathcal{A}'\}$, s.t. the set $\mathcal{A}_{H_0} = \{a \in \mathcal{A}' \mid \text{Supp}(a) = H_0\}$ is infinite. From the definition of $\mathcal{A}'$, one obtains that $\forall a, b \in \mathcal{A}_{H_0}$, $\text{Conc}(a) \not\equiv \text{Conc}(b)$. It is clear that $\forall a \in \mathcal{A}_{H_0}$, $\text{Conc}(a) \in \text{Cncs}(\Sigma)$. This implies that there are infinitely many different formulae having pairwise non-equivalent conclusions in $\text{Cncs}(\Sigma)$, formally, the set $\text{Cncs}(\Sigma)/\equiv$ is infinite, contradiction. This means that for every $H_0 \in \{\text{Supp}(a) \mid a \in \mathcal{A}'\}$, the set $\mathcal{A}_{H_0} = \{a \in \mathcal{A}' \mid \text{Supp}(a) = H_0\}$ is finite. □

Theorem 11 *Let $\mathcal{F} = (\text{Arg}(\Sigma), \mathcal{R}_{as})$ be an argumentation system built over a propositional knowledge base Σ, and $\mathcal{F}_{\downarrow} = (\text{Arg}(\Sigma)_{\downarrow}, \mathcal{R}_{as\downarrow})$ its sub-system. For all $a \in \text{Arg}(\Sigma)_{\downarrow}$, $\text{Status}(a, \mathcal{F}) = \text{Status}(a, \mathcal{F}_{\downarrow})$ under stable semantics.*

Proof. Let $S_1, \ldots, S_n$ be all the maximal consistent subsets of Σ. Since $(\text{Arg}(\Sigma)_{\downarrow}, \mathcal{R}_{\downarrow})$ and $(\text{Arg}(\Sigma), \mathcal{R})$ both contain at least one core of $(\text{Arg}(\Sigma)_{\downarrow}, \mathcal{R}_{\downarrow})$ (in fact, they both contain *all* cores of this set) then Lemma 23 implies that extensions of $(\text{Arg}(\Sigma), \mathcal{R})$ are exactly $\text{Arg}(S_i)$, and extensions of $(\text{Arg}(\Sigma)_{\downarrow}, \mathcal{R}_{\downarrow})$ are exactly $\text{Arg}(S_i) \cap \text{Arg}(\Sigma)_{\downarrow}$, when $1 \leq i \leq n$. Thus, the two frameworks have the same number of extensions and any argument of $\text{Arg}(\Sigma)_{\downarrow}$ is in the same number of extensions in them. Consequently, its status must be the same in both frameworks. □

Theorem 12 *Let $\mathcal{F} = (\text{Arg}(\Sigma), \mathcal{R}_{as})$ be an argumentation system built over a propositional knowledge base Σ. For all $a \in \text{Arg}(\Sigma) \setminus \text{Arg}(\Sigma)_{\downarrow}$, $\text{Status}(a, \mathcal{F}) = \text{Status}(b, \mathcal{F})$ where $b \in \text{Arg}(\Sigma)_{\downarrow}$ and $\text{Supp}(a) \approx \text{Supp}(b)$.*

Proof. Let $a \in \text{Arg}(\Sigma) \setminus \text{Arg}(\Sigma)_{\downarrow}$ and $b \in \text{Arg}(\Sigma)_{\downarrow}$ and let $\text{Supp}(a) \approx \text{Supp}(b)$. Since $\mathcal{R}_{as}$ satisfies $C1b$ and $C2b$ then in every complete labelling (Caminada, 2006a) a and b have the same label. This means that a and b belong to exactly the same stable extensions. Hence their status is the same. □

Theorem 13 *For every propositional knowledge base Σ, it holds that $|\text{Arg}(\Sigma)_{\downarrow}/\approx| \leq 2^n \cdot 2^{2^m}$, where $n = |\Sigma|$ and $m = |\text{Atoms}(\Sigma)|$.*

Proof. There are at most 2^n different supports of arguments. It is well-known that there are at most 2^{2^m} logically non-equivalent Boolean functions of m variables. Thus, for any support H, there are at most 2^{2^m} different non-equivalent arguments, where m is the number of different atoms in Σ. □

Theorem 14 *Let Σ be a propositional knowledge base and $\mathcal{F} = (\mathcal{A}, \mathcal{R}_{as})$ be an argumentation system such that $\mathcal{A} \subseteq \text{Arg}(\Sigma)_{\downarrow}$. Then every core of $\mathcal{F}$ is finite.*

Proof. Follows directly from Theorem 13 and Definition 14. □

Theorem 16 *Let $\mathcal{F}$ be an argumentation system built over a propositional knowledge base Σ using stable semantics and let $\mathcal{G}$ be one of its cores.* $\text{Output}_{sc}(\mathcal{F}) = \{x \in \mathcal{L}$ *s.t.* $\text{Output}_{sc}(\mathcal{G}) \vdash x\}$.

Proof. Let $\mathcal{G} = (\mathcal{A}_g, \mathcal{R}_g)$.
$\Rightarrow$ Let $h \in \text{Output}_{sc}(\mathcal{F})$. This means that $\exists a \in \mathcal{A}$ s.t. $a \in \text{Sc}(\mathcal{F})$ and $\text{Conc}(a) = h$. Let $a = (H, h)$ and let $H = \{f_1, \ldots, f_k\}$. Since a is an argument, then H is consistent and no formula in H can be deduced from other formulae in H. Then, $a' = (H, f_1 \wedge \ldots \wedge f_k)$ must also be an argument. Note that its conclusion contains only atoms from Σ, thus $a' \in \mathcal{A}'$. Consequently, there must exist an argument $a_g \in \mathcal{A}_g$ s.t. $a_g \approx_1 a'$. $\mathcal{G}$ is a core of $\mathcal{F}'$, thus they are equivalent w.r.t. EQ1b (Theorem 7). Since equivalent arguments have the same status in equivalent frameworks (Property 9) then a_g is sceptically accepted in $\mathcal{G}$. So, $\text{Output}_{sc}(\mathcal{G}) \vdash f_1 \wedge \ldots \wedge f_k$. Consequently, $\text{Output}_{sc}(\mathcal{G}) \vdash h$.

$\Leftarrow$ Let f be a propositional formula that can be deduced from $\text{Output}_{sc}(\mathcal{G})$. Let

$S_1, \ldots, S_n$ be all the maximal consistent subsets of Σ. According to Lemma 23, $\exists a \in \mathcal{A}_g$ s.t. $\text{Supp}(a) \subseteq S_1 \cap \ldots \cap S_n$ and $\text{Conc}(a) = f$. Let us denote $H = \text{Supp}(a)$. Obviously, $H \vdash f$. Furthermore, $H \subseteq S_1 \cap \ldots \cap S_n$. From those two facts, we conclude that it must exist an argument $a' \in \text{Arg}(\Sigma)$ s.t. $\text{Supp}(a') \subseteq H$ and $\text{Conc}(a') = f$. From Lemma 23, a' is sceptically accepted in $\mathcal{F}$. Thus, $f \in \text{Output}_{sc}(\mathcal{F})$. □

Theorem 18 *For every formula $\alpha \in \mathcal{L}_\Theta$, there exists $\sigma \in F_k^+$ s. t.* $\text{CN}(\alpha) = \text{CN}(\sigma)$.

Proof. By induction on the structure of formulas from $\mathcal{L}_\Theta$.
Base step. If α is an atomic formula, then $\alpha \in F_k^+$.
Induction step. Induction hypothesis: Assume that for each formula $\lambda \in \mathcal{L}_\Theta$ of depth less than n there exists $\mu \in F_k^+$ such that $\text{CN}(\lambda) = \text{CN}(\mu)$.
Consider $\alpha \in \mathcal{L}_\Theta$ whose depth is less than $n+1$. I.e., α is of the form $o_i(\gamma_1, \ldots, \gamma_{n_i})$ where every γ_h is of depth less than n. By the induction hypothesis, there exist $\gamma'_1, \ldots, \gamma'_{n_i}$ in F_k^+ such that $\text{CN}(\gamma_h) = \text{CN}(\gamma'_h)$ for $h = 1..n_i$. Equivalently, $m(\gamma_h) = m(\gamma'_h)$ for all m. As $(\mathcal{L}, \text{CN})$ is algebraic, $m(o(\gamma_1, \ldots, \gamma_{n_i})) = m(o(\gamma'_1, \ldots, \gamma'_{n_i}))$. There exists an absorption law that applies here because every γ'_h is in F_k^+. In symbols, $m(o(\gamma'_1, \ldots, \gamma'_{n_i})) = m(\delta')$ for some $\delta' \in F_k^+$. Therefore, there exists $\delta' \in F_k^+$ which is CN-equivalent to $o(\gamma'_1, \ldots, \gamma'_{n_i})$ hence CN-equivalent to α. □

Theorem 21 *Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ be an argumentation system built over a knowledge base Σ such that $\mathcal{R}$ satisfies C1b and C2b. If $\mathcal{F}$ contains a core of $\mathcal{G} = (\text{Arg}(\text{Base}(\mathcal{A})), \mathcal{R}(\mathcal{L})|_{\text{Arg}(\text{Base}(\mathcal{A}))})$, then for all $\mathcal{E} \subseteq \text{Arg}(\text{Base}(\mathcal{A}))$,*

- $\mathcal{F} \equiv_{EQ1b} \mathcal{F} \oplus \mathcal{E}$
- $\forall a \in \mathcal{A}, \text{Status}(a, \mathcal{F}) = \text{Status}(a, \mathcal{F} \oplus \mathcal{E})$
- $\forall e \in \mathcal{E} \setminus \mathcal{A}, \text{Status}(e, \mathcal{F} \oplus \mathcal{E}) = \text{Status}(a, \mathcal{F})$, *where $a \in \mathcal{A}$ is any argument s.t.* $\text{Supp}(a) \approx \text{Supp}(e)$.

Proof. Let $\mathcal{F}' = \mathcal{F} \oplus \mathcal{E}$ with $\mathcal{F}' = (\mathcal{A}', \mathcal{R}')$ and let $\mathcal{H} = (\mathcal{A}_h, \mathcal{R}_h)$ be a core of $\mathcal{G}$ s.t. $\mathcal{A}_h \subseteq \mathcal{A}$. We will first show that $\mathcal{H}$ is a core of both $\mathcal{F}$ and $\mathcal{F}'$.

- Let us first show that $\mathcal{H}$ is a core of $\mathcal{F}$. We will show that all conditions of Definition 14 are verified.
  - We have already seen why $\mathcal{A}_h \subseteq \mathcal{A}$.
  - We will show that $\forall a \in \mathcal{A}$, $\exists! a' \in \mathcal{A}_h$ s.t. $a' \approx a$. Let $a \in \mathcal{A}$. Since $a \in \mathcal{A}_g$ and $\mathcal{H}$ is a core of $\mathcal{G}$, then $\exists! a' \in \mathcal{A}_h$ s.t. $a' \approx a$.
  - Since $\mathcal{R} = \mathcal{R}(\mathcal{L})|_{\mathcal{A}}$ and $\mathcal{R}_h = \mathcal{R}(\mathcal{L})|_{\mathcal{A}_h}$ then from $\mathcal{A}_h \subseteq \mathcal{A}$ we obtain that $\mathcal{R}_h = \mathcal{R}|_{\mathcal{A}_h}$.

  Thus, $\mathcal{H}$ is a core of $\mathcal{F}$. Let us now show that $\mathcal{H}$ is also a core of $\mathcal{F}'$:
  - Since $\mathcal{A}_h \subseteq \mathcal{A}$ and $\mathcal{A} \subseteq \mathcal{A}'$ then $A_h \subseteq \mathcal{A}'$.
  - Let $a \in \mathcal{A}'$. Since $a \in \mathcal{A}_g$ and $\mathcal{H}$ is a core of framework $\mathcal{G}$, then $\exists! a' \in \mathcal{A}_h$ s.t. $a' \approx a$.
  - Since $\mathcal{R}' = \mathcal{R}(\mathcal{L})|_{\mathcal{A}'}$, $\mathcal{R}_h = \mathcal{R}(\mathcal{L})|_{\mathcal{A}_h}$ and $\mathcal{A}_h \subseteq \mathcal{A}'$, then we obtain that $\mathcal{R}_h = \mathcal{R}'|_{\mathcal{A}_h}$.

  We have shown that $\mathcal{H}$ is a core of $\mathcal{F}$ and of $\mathcal{F}'$. From Theorem 7, $\mathcal{F} \equiv_{EQ11} \mathcal{H}$ and $\mathcal{F}' \equiv_{EQ11} \mathcal{H}$. Since $\equiv_{EQ11}$ is an equivalence relation, then $\mathcal{F} \equiv_{EQ11} \mathcal{F}'$. Let $a \in \mathcal{A}$. From Property 9, $\text{Status}(a, \mathcal{F}) = \text{Status}(a, \mathcal{F}')$.

  Let $e \in \mathcal{A}' \setminus \mathcal{A}$ and let $a \in \mathcal{A}$ be an argument such that $\text{Supp}(a) \approx \text{Supp}(e)$. Since a and e are attacked by the same arguments, they are in the same complete labellings (Caminada, 2006a); thus they are in the same extensions. Consequently, they have the same status: $\text{Status}(e, \mathcal{F}') = \text{Status}(a, \mathcal{F}')$. Since we have seen that $\text{Status}(a, \mathcal{F}') = \text{Status}(a, \mathcal{F})$, then $\text{Status}(e, \mathcal{F}') = \text{Status}(a, \mathcal{F})$.

□

Theorem 22 *Let* $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ *be an argumentation system built over a knowledge base* Σ *and let* $\mathcal{E} \subseteq \mathcal{A}$*. If* $\mathcal{F} \ominus \mathcal{E}$ *contains a core of* $\mathcal{G} = (\mathrm{Arg}(\mathrm{Base}(\mathcal{A})), \mathcal{R}(\mathcal{L})|_{\mathrm{Arg}(\mathrm{Base}(\mathcal{A}))})$*, then:*

- $\mathcal{F} \equiv_{EQ1b} \mathcal{F} \ominus \mathcal{E}$
- $\forall a \in \mathcal{A} \setminus \mathcal{E}$, $\mathrm{Status}(a, \mathcal{F}) = \mathrm{Status}(a, \mathcal{F} \ominus \mathcal{E})$.

Proof. This result is a consequence of Theorem 21. □