Multifractal characterization for bivariate data

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Abstract — Multifractal analysis is a reference tool for the analysis of data based on local regularity. Historically fundamentally univariate, recent contributions have explored a first multivariate theoretical grounding for multifractal analysis and shown that it can capture **transient higher-order dependence beyond correlation between time series**. This work studies the use of a quadratic model for the joint multifractal spectrum of bivariate time series and provides expressions for the Pearson correlation in terms of random walk and multifractal cascade dependence parameters that quantify such non-linear, higher-order dependencies.

Multifractal analysis

Multifractal Spectrum

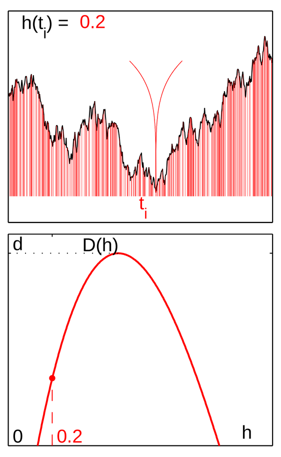
- Local Regularity:
  - Regularity of function $X(t)$ at t: regularity exponent
  - Most common: *Hölder exponent* $h(t) \geq 0$

$$|X(t+a) - X(t)| \overset{a \to 0}{\sim} |a|^{h(t)}$$

- Multifractal Spectrum:

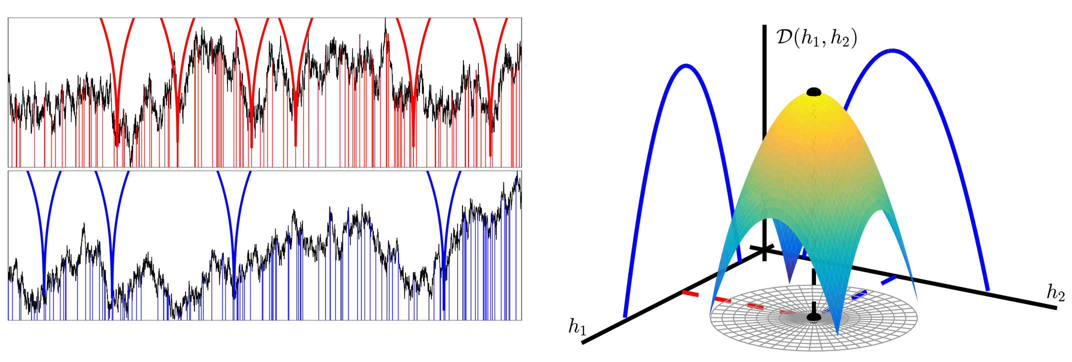
$$\mathcal{D}(h) = \dim_{\text{Hausdorff}}\{t : h(t) = h\}$$

$\longrightarrow$ "Amount" of points with given regularity



- Bivariate Multifractal Spectrum:
  - Bivariate signal: $\mathbf{X} = (X_1, X_2)$
  - Hölder exponents: $(h_1(t), h_2(t))$

$$\mathcal{D}(h_1, h_2) = \dim_{\text{Hausdorff}}\{t : h_1(t) = h_1 \text{ and } h_2(t) = h_2\}$$



- **Problem:** Can not be computed in practice
  $\longrightarrow$ Use *multifractal formalism* to estimate

Wavelet leaders

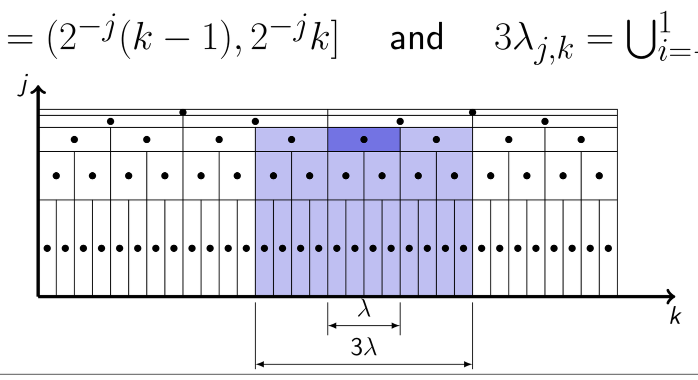
- *Discrete wavelet transform* (scale 2^{-j}, position $k = t\,2^j$)

$$X(t) \longrightarrow d_X(j,k) = d_X(\lambda)$$

- *Wavelet leaders* (Wendt, Abry & Jaffard, 2007)

$$L_X(j,k) = \sup_{\lambda' \in 3\lambda} |c_{\lambda'}|$$

$$\lambda = \lambda_{j,k} = (2^{-j}(k-1), 2^{-j}k] \quad \text{and} \quad 3\lambda_{j,k} = \cup_{i=-1}^{1} \lambda_{j,k+i}$$



Multifractal Formalism

1. Structure functions S and scaling exponents ζ

$$S(q_1,q_2,j) = \frac{1}{n_j}\sum_{k=1}^{n_j} L_{X_1}(j,k)^{q_1} L_{X_2}(j,k)^{q_2} \sim 2^{j\zeta(q_1,q_2)}, \quad 2^j \to 0$$

2. Legendre spectrum $\mathcal{L}$: *Upper bound estimate* for $\mathcal{D}$

$$\mathcal{L}(h_1,h_2) = \inf_{q_1,q_2} \left(1 + q_1 h_1 + q_2 h_2 - \zeta(q_1,q_2)\right) \geq \mathcal{D}(q_1,q_2).$$

$\longrightarrow$ Robust and easy to compute in practice

Quadratic approximation of $\mathcal{L}(h_1,h_2)$

- Bivariate cumulants of $(\ln L_{X_1}(j,k), \ln L_{X_2}(j,k))$

$$C_{p_1p_2}(j) = \mathbb{E}[\ln(L_{X_1}(j,\cdot))^{p_1} \ln(L_{X_2}(j,\cdot))^{p_2}], \quad p_1+p_2 \geq 1$$
$$= c^0_{p_1p_2} + c_{p_1p_2}\ln 2^j$$

- Cumulant expansion of bivariate spectrum $(b = c_{20}c_{02} - c_{11}^2 \geq 0)$

$$\mathcal{L}(h_1,h_2) \approx 1 + \frac{c_{02}b}{2}\left(\frac{h_1 - c_{10}}{b}\right)^2 + \frac{c_{20}b}{2}\left(\frac{h_2 - c_{01}}{b}\right)^2 - c_{11}b\left(\frac{h_1 - c_{10}}{b}\right)\left(\frac{h_2 - c_{01}}{b}\right)$$

- Interpretation:
  - c_{01}, c_{10}: average regularity on each component
  - c_{02}, c_{20}: width of regularity fluctuations on each component
  - c_{11}: leading-order joint regularity fluctuation
- Estimation $\longrightarrow$ linear regressions $C_{p_1p_2}(j)$ *vs* $\ln 2^j$

Parametrization in natural coordinates

- Level sets of $\mathcal{L}(\mathbf{h})$: rotated / translated ellipses in (h_1,h_2) plane
- *Natural parameters*:
  center $\mathbf{h}^m$, rotation θ, major&minor half-axes α_1&α_2, eccentricity ϵ

$$\theta = \frac{1}{2}\arctan\left(\frac{2c_{11}}{c_{20} - c_{02}}\right),$$

$$\alpha_1 = 2\sqrt{\frac{c_{20}c_{02} - c_{11}^2}{-\sqrt{(c_{02}-c_{20})^2 + 4c_{11}^2} - c_{02} - c_{20}}},$$

$$\alpha_2 = 2\sqrt{\frac{c_{20}c_{02} - c_{11}^2}{+\sqrt{(c_{02}-c_{20})^2 + 4c_{11}^2} - c_{02} - c_{20}}}$$

$$\epsilon = \sqrt{\alpha_1 - \alpha_2} = \frac{1}{2}\left(\sqrt{\gamma - c_{02} - c_{20}} - \sqrt{-\gamma - c_{02} - c_{20}}\right),$$

$$\text{where } \gamma \triangleq \sqrt{(c_{02} - c_{20})^2 + 4c_{11}^2},$$

Multifractal Model Process

Bivariate Multifractal Random Walk

Synthetic process with bivariate multifractal behavior

- Definition
  - Use **two pairs** of stochastic processes
  - **Pair 1:** bivariate fractional Gaussian noise $G_1(t), G_2(t)$
    - Self-similarity parameters: H_1, H_2
    - Covariance matrix:

$$\Sigma_{ss} = \begin{pmatrix} 1 & \rho_{ss} \\ \rho_{ss} & 1 \end{pmatrix}$$

  - **Pair 2:** Gaussian processes $\omega_1(t), \omega_2(t)$
    - Multifractality parameters: λ_1, λ_2
    - Covariance function: Σ_{mf} such that

$$\{\Sigma_{mf}\}_{ij}(k,l) = \boldsymbol{\rho_{mf}}(i,j)\lambda_i\lambda_j \log\left(\frac{N}{|k-l|+1}\right), \; i = 1,2$$

where

$$\boldsymbol{\rho_{mf}} = \begin{pmatrix} 1 & \rho_{mf} \\ \rho_{mf} & 1 \end{pmatrix}$$

$\longrightarrow$ Logarithmic covariance to induce multifractality

  - $G_i(t), \omega_i(t)$ synthesized following (Helgason, Pipiras & Abry, 2011)
  - **Final process**: bivariate MRW

$$X_i(t) = \sum_{k=1}^{t} G_i(k)e^{\omega_i(k)}, \quad i = 1,2$$

- Multifractal Properties
  - Cumulants:
    - $c_{10} = H_1 + \lambda_1^2/2$ and $c_{01} = H_2 + \lambda_2^2/2$
    - $c_{20} = -\lambda_1^2$ and $c_{02} = -\lambda_2^2$
    - $c_{11} = -\rho_{mf}\lambda_1\lambda_2$

Correlation and dependence

- Correlation coefficient of bivariate MRW:

$$\rho_{bMRW} = \rho_{ss} \cdot f(\rho_{mf}, \lambda_1, \lambda_2)$$

$\longrightarrow$ Can have $\rho_{bMRW} = 0$ with $\rho_{mf} \neq 0$!

- From elementary properties of log-normal random variables:

$$\rho_{bMRW} = \rho_{ss} \cdot e^{\left(\rho_{mf}\lambda_1\lambda_2 - \frac{1}{2}(\lambda_1^2 + \lambda_2^2)\right)\log(N)}$$

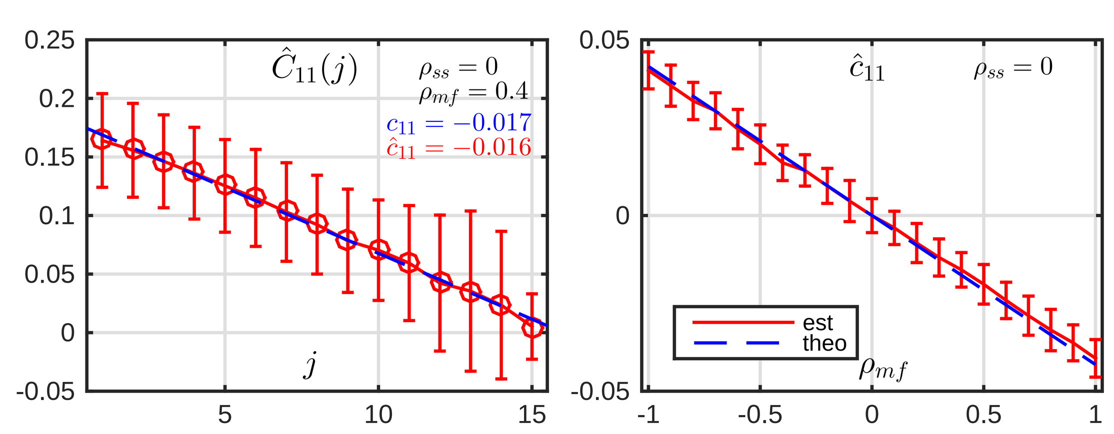
$\longrightarrow$ ρ_{ss} and ρ_{mf}: expansion coefficients for joint dependence of X

- Natural estimators:

$$\hat{\rho}_{mf} \triangleq -\hat{c}_{11}/\sqrt{|\hat{c}_{20}\hat{c}_{02}|}$$
$$\hat{\rho}_{ss} \triangleq \hat{\rho}_{bMRW} e^{\left(\hat{c}_{11} - \frac{1}{2}(\hat{c}_{20} + \hat{c}_{02})\right)\log(N)}$$

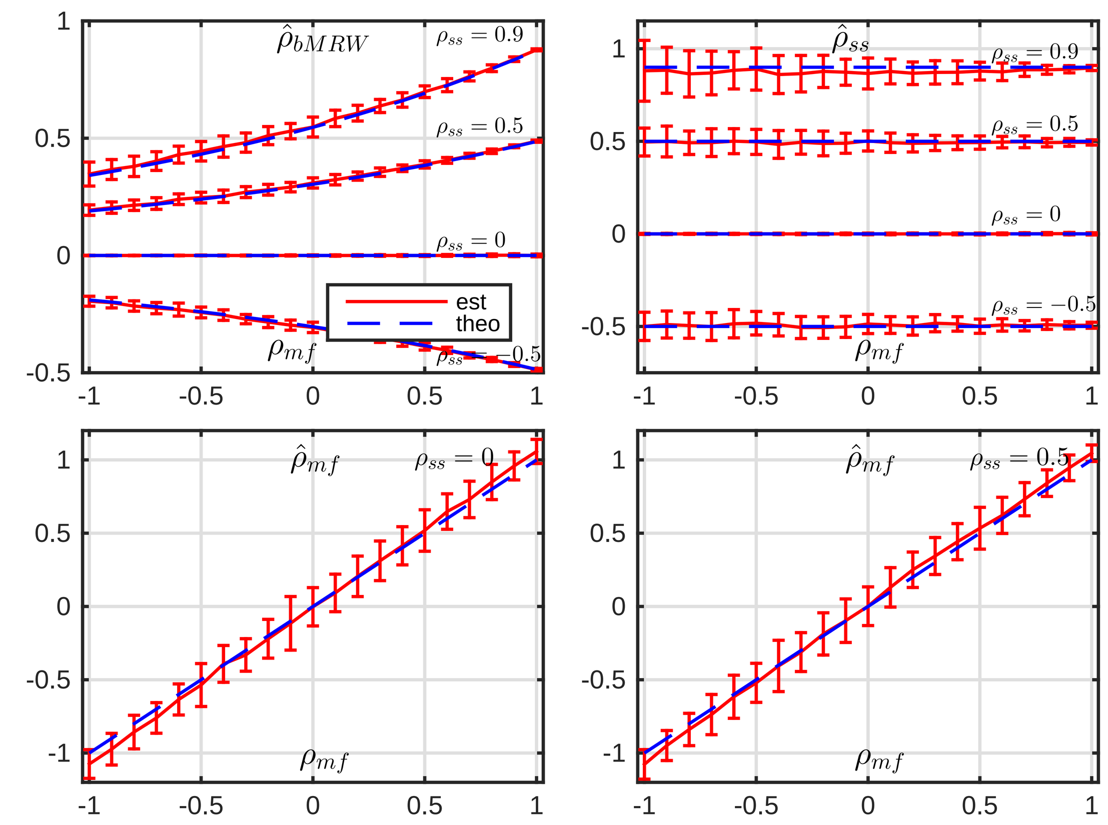
Estimation performance

$C_{11}(j)$ and c_{11}



- $\hat{C}_{11}(j)$ linear in j, $\hat{c}_{11}$ excellent for all ρ_{mf}

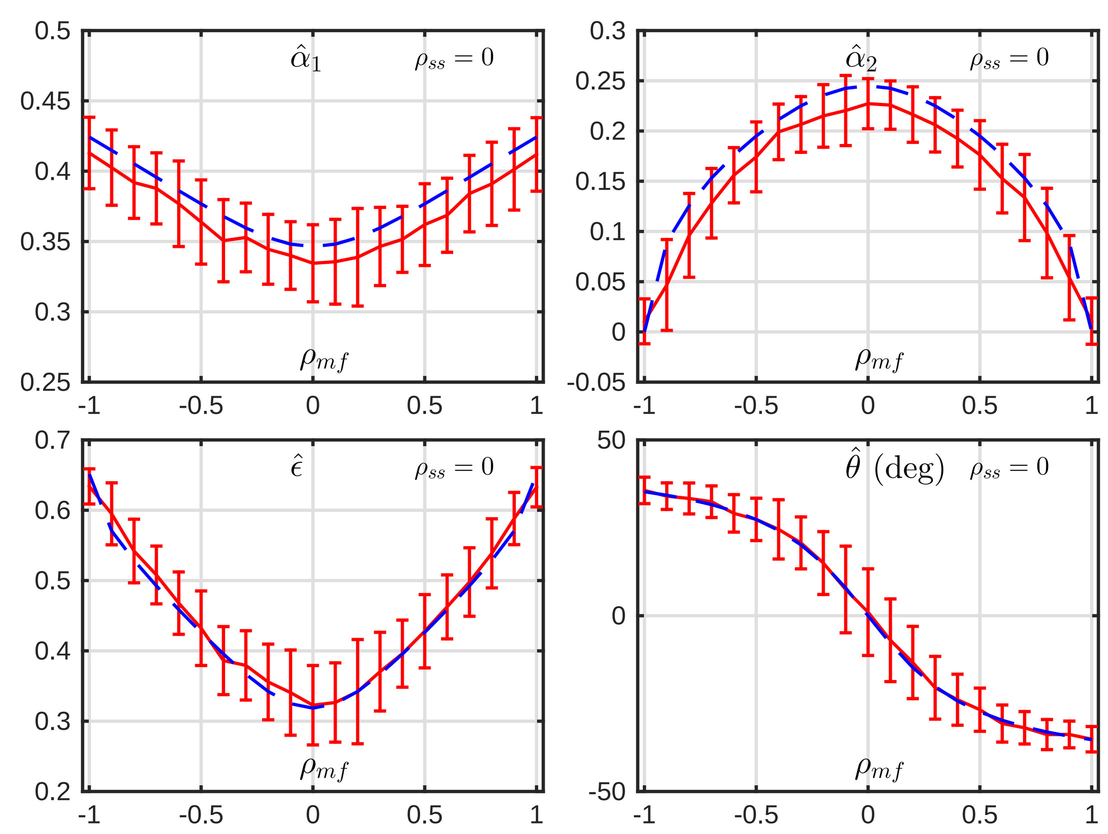
Dependence parameters



- Relevant model and excellent estimation performance
- $\rho_{mf} \neq 0 \implies$ dependence beyond correlation

Natural bivariate multifractal parameters

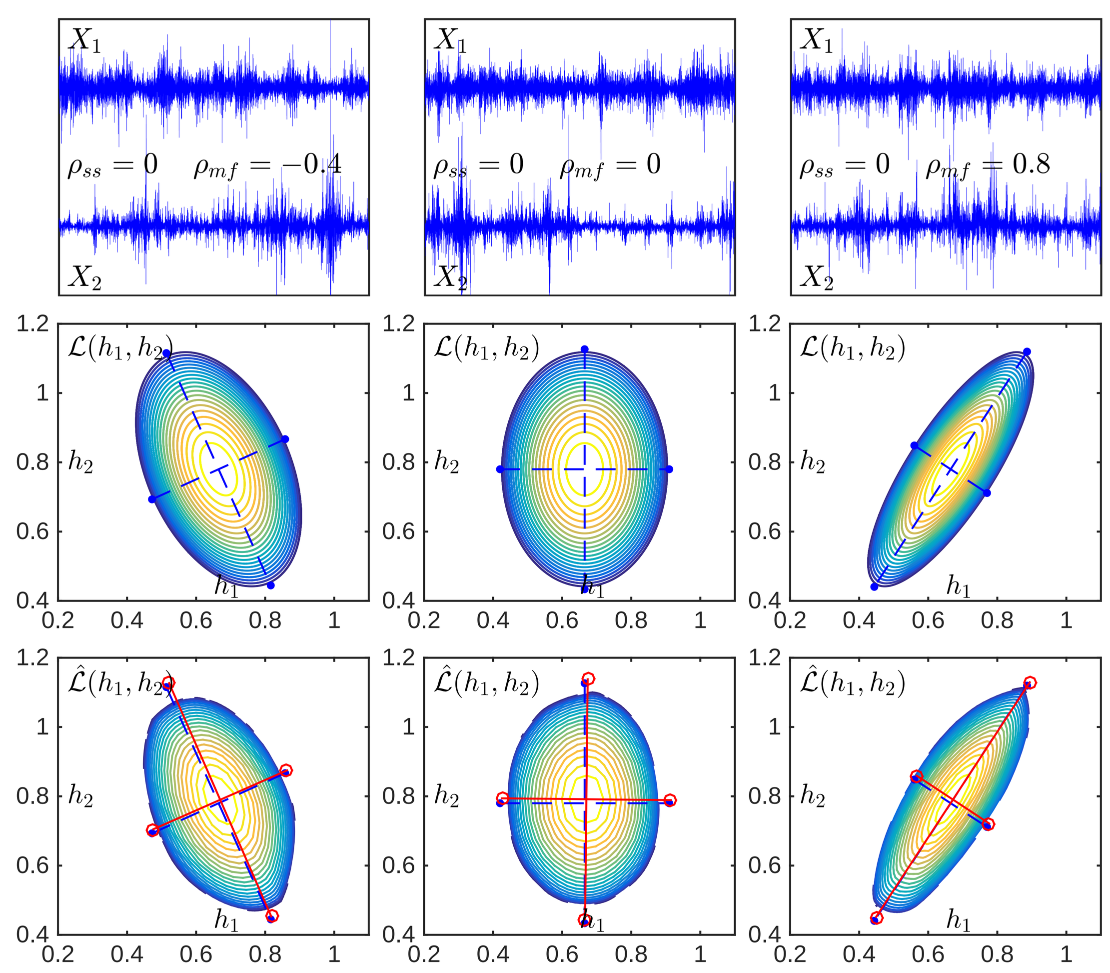
- Set $\rho_{ss} = 0 \implies \rho_{bMRW} = 0$



- Robust estimation of quadratic multifractal model parameters
- Higher order statistical dependence summarized in natural shape & orientation parameters of spectra

Legendre spectra

- Set $\rho_{ss} = 0 \implies \rho_{bMRW} = 0$



- Spectra caputre higher-order statistical dependence beyond Pearson correlation
- $\rho_{ss} = 0, \rho_{mf} = 0 \implies$: uncorrelated, independent
- $\rho_{ss} = 0, \rho_{mf} > 0 \implies$: uncorrelated, positive dependence
- $\rho_{ss} = 0, \rho_{mf} < 0 \implies$: uncorrelated, negative dependence
- natural parameter estimates more accurate for support of $\mathcal{L}(h_1, h_2)$

References

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