Bootstrap for Log Wavelet Leaders Cumulant based Multifractal Analysis

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EUSIPCO 2006, Firenze, 07.09.2006

Motivation

- Multifractal Analysis (MFA)
  - Scaling in data: numerous applications of very different nature
  - Usually based on moments of wavelet coefficients
- Wavelet Leaders:
  significant theoretical/practical qualities
- Log-cumulants based MFA:
  emphasizes difference mono- & multi-fractal processes

Goal:

Practical procedure for obtaining accurate log-cumulant estimates and confidence intervals from one single realization of data

→ Do Wavelet Leaders improve current estimation procedures?

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Wavelet Coefficients and Wavelet Leaders

- Discrete Wavelet Coefficients

$$\begin{aligned} \psi_{j,k}(t) &= 2^{-j}\psi_0(2^{-j}t-k) \quad \text{dyadic grid} \\ d_X(j,k) &= \langle \psi_{j,k} | X \rangle \end{aligned}$$

- Wavelet Leaders $\quad L_X(j,k) = \sup_{\lambda' \subset 3\lambda_{j,k}} |d_{\lambda'}|$

$$\lambda_{j,k} = [k2^j, (k+1)2^j), \qquad 3\lambda_{j,k} = \lambda_{j,k-1} \cup \lambda_{j,k} \cup \lambda_{j,k+1}$$

Supremum:
taken on $d_X(j,k)$,
in time neighborhood $3\lambda_{j,k}$,
over all finer scales $2^{j'} < 2^j$

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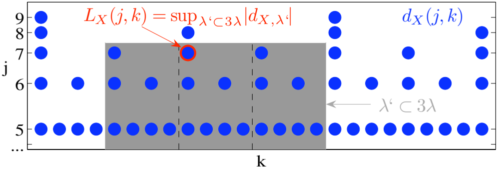

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Scaling and Multifractal

- Scale Invariance:

$$\frac{1}{n_j}\sum_{k=1}^{n_j} L_X(j,k)^q = F_q|a|^{\zeta(q)} \tag{1}$$

for $a \in [a_m, a_M]$, $a_M/a_m >> 1 \quad (a = 2^j)$

- Multifractal Analysis:

$\zeta(q) \longrightarrow$ singularity spectrum $D(h)$

- Scaling exponent $\zeta(q)$:

$\zeta(q) = qH$ linear $\rightarrow X$ monofractal
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Log-Cumulants

Some classes of processes:

- Eq. (1) $\longrightarrow$ $\mathbb{E} L_X(j,\cdot)^q = F_q |2^j|^{\zeta(q)}$ (2)
- $\ln \mathbb{E} e^{q \ln L_X(j,\cdot)} = \sum_{p=1}^{\infty} C_p^j \frac{q^p}{p!}$

  C_p^j - cumulants of random variable $\ln L_X(j,\cdot)$

  $C_p^j = c_p^0 + c_p \ln 2^j, \quad \forall p \geq 1$ (3)

- Eqs. (2) + (3):

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- Measurement of $\zeta(q)$ replaced by those of log-cumulants c_p:

$$\zeta(q) = c_1 q + c_2 \frac{q^2}{2} + c_3 \frac{q^3}{6} + \cdots$$

  - X monofractal: $\zeta(q)$ linear
    $\Rightarrow \forall p > 1 : \; c_p \equiv 0$
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Estimating the log-cumulants c_p

1. At each scale j:
   a - compute n_j Leaders $L_X(j,k)$
   b - estimate cumulants $\hat{C}_p^j$ of $\ln L_X(j,\cdot)$
2. $\hat{c}_p$: $C_p^j = c_p^0 + c_p \ln 2^j$ $\rightarrow$ linear regressions $\hat{C}_p^j$ vs. $\ln 2^j$

$$\hat{c}_p = \log_2 e \sum_{j=j_1}^{j_2} w_j \hat{C}_p^j$$

Weights w_j: reflect confidence granted to each $\hat{C}_p^j$
here $w_j = 1/n_j$

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Resampling and Bootstrap Estimates

Non parametric Bootstrap:

Statistical properties of estimate from single realization by repeated resampling from available data

At each scale j: $\mathcal{L}_j = \{L_X(j,1), \cdots, L_X(j,n_j)\}$

- $\mathcal{L}_j \rightarrow$ estimates $\hat{C}_p^j$, $\hat{c}_p$
- $\mathcal{L}_j \rightarrow R$ bootstrap resamples $\mathcal{L}_j^{*(1)}, \cdots, \mathcal{L}_j^{*(R)}$:

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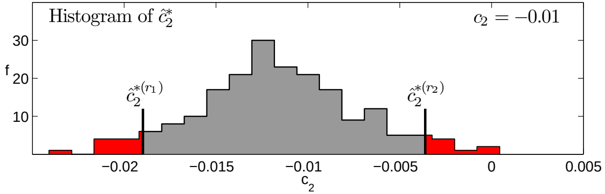
Bootstrap Confidence Intervals



- $100(1-\alpha)\%$ confidence intervals for c_ps:

$$\mathrm{CI}_p = \left(Q_p\left(\frac{\alpha}{2}\right), Q_p\left(1-\frac{\alpha}{2}\right)\right) = \left(\hat{c}_p^{*(r_1)}, \hat{c}_p^{*(r_2)}\right)$$

- $Q_p(\alpha)$ - α-th quantile of empirical distribution $\hat{c}_p^*$:

$$r_1 = \lfloor \frac{R\alpha}{2} \rfloor \text{ and } r_2 = R - r_1 + 1$$

Monte Carlo Simulation

- We have now: $\hat{c}_p,\ CI \quad \rightarrow \quad$ Performance ?
- Apply procedures to large number N_{MC} of realizations of synthetic multifractal processes with known scaling properties.
- Scaling Processes with stationary increments:
  - Fractional Brownian Motion (FBM):
    only Gaussian exactly selfsimilar process
    $\zeta(q) = qH,\ c_1 = H,\ p \geq 1:\ c_p \equiv 0 \leftarrow$ monofractal
  - Multifractal Random Walk (MRW):
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Statistical Performance: MSE

$$\text{MSE} = \sqrt{\text{Var}_{\text{MC}}\{\hat{c}_p\} + (c_p - \mathbb{E}_{\text{MC}}\{\hat{c}_p\})^2}$$

MSE×10^3	FBM				
	c_1	c_2	c_3	c_4	c_5
Coefficients	15.5	37.3	187.7	1251	9803
Leaders	**10.8**	**4.1**	**1.8**	**1.0**	**0.7**

MSE×10^3	MRW				
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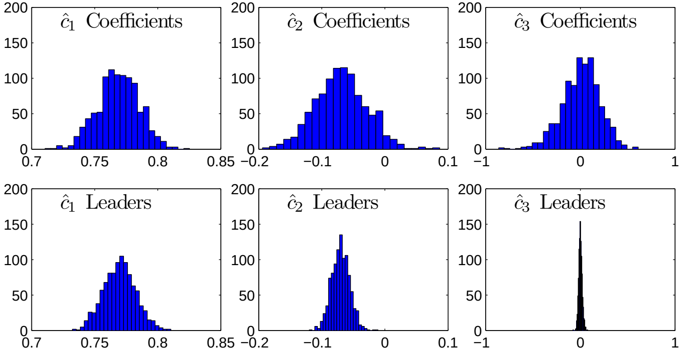
Statistical Performance continued

Monte Carlo empirical distributions of estimates:



Confidence Intervals by Bootstrap

Nominal Coverage: 95%

$$\text{Empirical Coverage} = \frac{\#(c_p \in CI_p)}{N_{\text{MC}}} \quad \text{(bias corrected)}$$

Empirical Coverage (in %)	FBM				
	c_1	c_2	c_3	c_4	c_5
Coefficients	92.1	92.3	91.3	87.8	87.7
Leaders	83.4	90.3	94.7	97.2	98.0

Empirical Coverage (in %)	MRW				
	c_1	c_2	c_3	c_4	c_5
Coefficients	98.6	95.5	92.6	90.3	90.8
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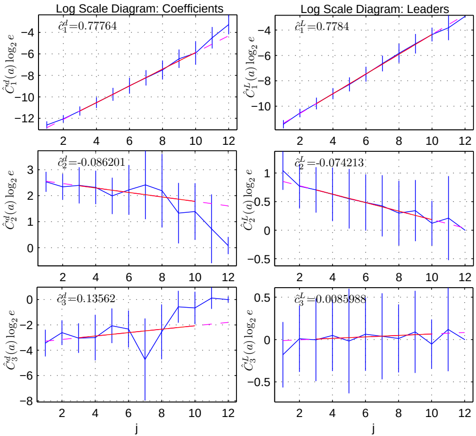
Practical Procedure

Coefficients Leaders



$$C_p^j = c_p^0 + c_p \ln 2^j$$

From single realization:

- Estimates $\hat{c}_p$
- Bootstrap CI for c_p
- Bootstrap CI for C_p^j:
  $\longrightarrow$ regression range

Conclusions and Perspectives

- Leaders based estimation procedure significantly outperforms Coefficients based one
- Bootstrap provides highly relevant confidence intervals for log-cumulants c_p
- Practical procedure that can be applied to a single, finite sample of empirical data
- Perspectives:
  - Advanced bootstrap techniques (pivoting, adjusted limits, ...)
  - Bootstrap hypothesis tests on c_p: testing mono-vs. multifractal

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Multifractal Analysis

Multifractal Analysis:

- Local regularity of X at t: Hölder exponent $h(t)$
  X is $C^{\alpha}(t_0)$ if $\quad \exists C, \alpha > 0;\ P_{t_0}(t);\ deg(P_{t_0}) < \alpha :$
  $$|X(t) - P_{t_0}(t)| < C|t - t_0|^{\alpha}$$
  $\rightarrow\ h(t_0) = \sup_{\alpha}\{\alpha : X \in C^{\alpha}(t_0)\}$
- Singularity spectrum $D(h)$:
  Haussdorf dimensions of $\{t_i | h(t_i) = h\}$

Empirical Multifractal Analysis:

$D(h)$ obtained as Legendre transform of estimates of $\zeta(q)$