TESTING MONO-VS. MULTIFRACTAL WITH BOOTSTRAPPED WAVELET LEADERS

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ABSTRACT

In many applications where data possess scaling properties, it is of importance to decide whether the data are better modelled with mono- or multifractal processes. However, so far no appropriate test is available. For this purpose, we propose here to test, using a bootstrap procedure, whether the second cumulant of the log of the wavelet coefficients or wavelet Leaders of the data is zero. We study the p-value and thepower of the tests through numerical simulation, using synthetic multifractal processes, and end up with a powerful procedure for practically discriminating mono- vs. multifractal processes.

1. MULTIFRACTAL, LEADERS AND CUMULANTS

Wavelet Leaders. Let X be the process under investigation, and n its observation duration. Let us denote by $d_X(j,k) = \langle \psi_{j,k} | X \rangle$ its wavelet coefficients at scales 2^j and time positions $2^j k$. Let us introduce the indexing $\lambda_{j,k} = [k2^j, (k+1)2^j)$ and the union $3\lambda_{j,k} = \lambda_{j,k-1} \cup \lambda_{j,k} \cup \lambda_{j,k+1}$. The *wavelet leaders* $L_X(j,k)$ are defined as $L_X(j,k) = \sup_{\lambda' \subset 3\lambda_{j,k}} |d_{\lambda'}|$, where the supremum is taken on the $d_X(\cdot,\cdot)$ in the time neighborhood $3\lambda_{j,k}$ over all finer scales $2^{j'} < 2^j$ (cf. [1]).

Scaling and Multifractal. A processes X is said to possess *scaling* properties if, for some $q \in [q_*^-, q_*^+]$, the time averages of $|L_X(j,k)|^q$ taken at fixed scales display power law behaviors with respect to scales, $\langle |L_X(j,\cdot)|^q \rangle = F_q |2^j|^{\zeta(q)}$, over a wide range of scales (equiv. for $|d_X(j,\cdot)|^q$). The $\zeta(q)$ are referred to as the scaling exponents of X and are closely related to its multifractal spectrum. The process X is said to be *monofractal* when $\zeta(q)$ is linear in q, i.e. $\zeta(q) = qH$, and *multifractal* when $\zeta(q) \neq qH$ (cf. [1] and references therein).

Log-Cumulants. Through a second characteristic function type expansion argument, scaling implies that $\zeta(q) = \sum_{p=1}^{\infty} c_p \frac{q^p}{p!}$ and $C_p^j = c_p^0 + c_p \ln 2^j$, where the C_p^j stand for the cumulants of order $p \geq 1$ of the random variable $\ln |L_X(j,\cdot)|$ (or equiv. $\ln |d_X(j,\cdot)|$ [3]). Thus, the measurements of the scaling exponents $\zeta(q)$ can be replaced by those of the *log-cumulants* c_p, emphasizing the difference between monofractal ($\forall p \geq 2 : \;\; c_p \equiv 0$) and multifractal processes. Estimates $\hat{c}_p$ of the log-cumulants c_p are obtained by linear regression of $\hat{C}_p^j$ vs. j.

2. BOOTSTRAP SIGNIFICANCE TEST

We want to test $H_{\text{null}} : c_{2,\text{null}} \equiv 0$ using a simple test statistic $T = c_2 - c_{2,\text{null}}$, with the observed value denoted by $t = \hat{c}_2 - c_{2,\text{null}}$. At each scale j, R simulated samples $\{L_X^*(j,\cdot)\}$ ($\{d_X^*(j,\cdot)\}$) of length n_j are generated from the original sample $\{L_X(j,\cdot)\}$ ($\{d_X(j,\cdot)\}$) of length n_j using a moving blocks bootstrap. From these resamples, the R bootstrap replica $\hat{C}_2^{j*}$ and $\hat{c}_2^*$ are estimated, and the simulated values $t^* = \hat{c}_2^* - \hat{c}_2$ are calculated and used to estimate the cdf $\hat{F}_{\text{null}}(\tau) = \frac{\#\{t^* \leq \tau\}}{R}$. The p-value, $p = Pr(T \geq t | H_{\text{null}})$, is now approximated by the bootstrap p-value, $p^* = Pr^*(t^* \geq t | \hat{F}_{\text{null}}) = \frac{\#\{t^* \geq t\}}{R}$ (cf. [2]).

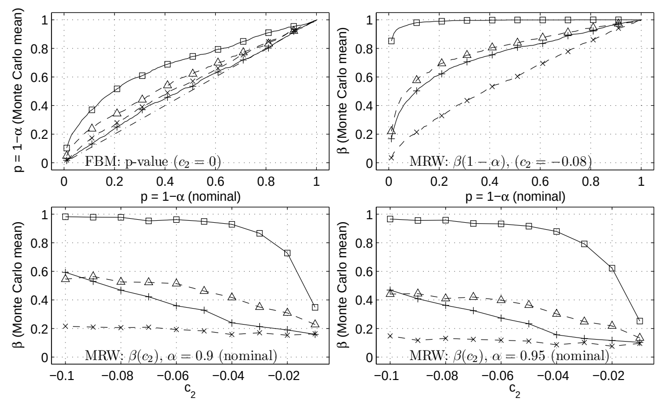

Fig. 1. Empirical p-value (top left) and power β (top right) vs. nominal $p = 1-\alpha$. Power $\beta(c_2)$ for nominal $\alpha = 0.9$ (bottom left) and $\alpha = 0.95$ (bottom right). The symbols ($\triangle$; $\square$; $\times$; $+$) correspond to ($L_X, n = 2^{12}$; $L_X, n = 2^{15}$; $d_X, n = 2^{12}$; $d_X, n = 2^{15}$).

3. RESULTS

We use a large number of realizations of (monofractal) fractional Brownian motion (FBM) to determine the relevance of p^*, and of Multifractal random walk (MRW) to evaluate the power β of the test. The results are summarized in Fig. 1: Whereas the d_X based tests have p^* closer to nominal p than on L_X based tests (top left), the latter have significantly larger power than the former (top right and bottom row). This is partly due to the fact that $\hat{c}_p$ based on L_X possess smaller variance. Most important, the tests involving L_X maintain large power ($\beta > 0.8$ for $n = 2^{15}$) over a wide range of values for c_2, including c_2 close to zero, for usual α, whereas the power of tests using d_X is, in comparison, poor (bottom row). We conclude that with the Leaders-based procedure described here, a powerful test is available for practically discriminating monofractal vs. multifractal processes.

4. REFERENCES

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