On the impact of the number of vanishing moments on the dependence structures of compound Poisson motion and fractional Brownian motion in multifractal time

B. Vedel, H. Wendt, P. Abry, S. Jaffard

Abstract From a theoretical perspective, scale invariance, or simply scaling, can fruitfully be modeled with classes of multifractal stochastic processes, designed from positive multiplicative martingales (or cascades). From a practical perspective, scaling in real-world data is often analyzed by means of multiresolution quantities. The present contribution focuses on three different types of such multiresolution quantities, namely increment, wavelet and Leader coefficients, as well as on a specific multifractal processes, referred to as Infinitely Divisible Motions and fractional Brownian motion in multifractal time. It aims at studying, both analytically and by numerical simulations, the impact of varying the number of vanishing moments of the mother wavelet and the order of the increments on the decay rate of the (higher order) covariance functions of the (q-th power of the absolute values of these) multiresolution coefficients. The key result obtained here consist of the fact that, though it fastens the decay of the covariance functions, as is the case for fractional Brownian motions, increasing the number of vanishing moments of the mother wavelet or the order of the increments does not induce any faster decay for the (higher order) covariance functions.

B. Vedel
Université de Bretagne Sud, Université Européenne de Bretagne, Campus de Tohannic, BP 573, 56017 Vannes, e-mail: beatrice.vedel@univ-ubs.fr, this work has mostly been completed while B. Vedel was at ENS Lyon, on a Post-Doctoral Grant supported by the gratefully acknowledged *Foundation Del Duca, Institut de France, 2007*

H. Wendt, P. Abry
ENSLyon, CNRS UMR 5672, 46 allée d'Italie, 69364 Lyon cedex, e-mail: herwig.wendt@ens-lyon.fr, patrice.abry@ens-lyon.fr

S. Jaffard
Université Paris Est, CNRS, 61, avenue du Général de Gaulle, 94010 Créteil, e-mail: jaffard@univ-paris12.fr

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1 Motivation

Scale invariance. Scale invariance, or simply scaling, is a paradigm nowadays commonly used to model and describe empirical data produced by a large variety of different applications [29]. Scale invariance consists of the idea that the data being analyzed do not possess any characteristic scale (of time or space). Often, instead of being formulated directly on the data $X(t)$, scale invariance is expressed through *multiresolution coefficients*, $T_X(a,t)$, such as increments or wavelet coefficients, computed from the data, and depending jointly on the position t and on the analysis scale a. Practically, scaling is defined as power law behaviors of the (q-th order) moments of the (absolute values of the) $T_X(a,t)$ with respect to the analysis scale a:

$$\frac{1}{n_a}\sum_{k=1}^{n_a} |T_X(a,ak)|^q \simeq c_q a^{\zeta(q)}, \tag{1}$$

for a given range of scales $a \in [a_m, a_M]$, $a_M/a_m \gg 1$, and for some (statistical) orders q.

Performing scaling analyses on data mostly amounts to testing the adequacy of Eq. (1) and to estimate the corresponding scaling exponents $\zeta(q)$.

Fractional Brownian motion and vanishing moments. Fractional Brownian motion (fBm) has been amongst the first stochastic process used to model scaling properties in empirical data and still serves as the central reference. It consists of the only Gaussian self similar process with stationary increments, see [23] and references therein. It is mostly controlled by its self-similarity parameter, $0 < H < 1$, and its scaling exponents behave as a linear function of q: $\zeta(q) = qH$.

For $1/2 < H < 1$, the increment process of fBm is characterized by a so-called *long range dependence* structure, or *long memory* [23, 6]. This property significantly impairs the accurate estimation of the process parameters.
Two seminal contributions [12, 13] (see also [25]) showed that the wavelet coefficients of fBm are characterized by a *short range* correlation structure, as soon as the number N_ψ of vanishing moments (cf. Section 3, for definition) of the analyzing mother wavelet ψ_0 is such that:

$$N_\psi \geq 2H + 1. \tag{2}$$

This *decorrelation* or *whitening* property of the wavelet coefficients implies, because of the Gaussian nature of fBm, that the entire dependence structure is turned *short range*. This significantly simplifies the estimation of the model parameters, and notably that of the self-similarity parameter H [2, 26, 30, 10]. It has, later, been shown that this decorrelation property is also effective when using higher order increments (increments of increments...) [16], which equivalently amounts to increasing the number of vanishing moments of the mother wavelet (cf. Section 3). This key role of N_ψ, together with the possibility that it can be easily tuned by practitioners, has constituted the fundamental motivation for the systematic and popular use of wavelet transforms for the analysis of empirical data which are likely to possess scaling properties.

Multiplicative martingales and vanishing moments. Often in applications, scaling exponents are found to depart form the linear behavior qH associated to fBm. To account for this, following Mandelbrot's seminal works (e.g., [20]), multiplicative cascades (or more technically multiplicative martingales) have received considerable interests in modeling scaling in applications. Notably, they are considered as reference processes to model multifractal properties in data, a particular instance of scaling. Often, it has been considered heuristically by practitioners that the benefits of conducting wavelet analyses over such models were equivalent to those observed when analyzing fBm. Despite its being of crucial importance, this issue received little attention at a theoretical level (see a contrario [15, 3]). This can partly be explained by the fact that Mandelbrot's multiplicative cascades, that remain up to the years 2000, the most (if not the only) practically used process to model multifractal scaling properties in data, present involved statistical characteristics. Hence, the derivation of the dependence structure of their wavelet coefficients has been considered difficult to obtain analytically. In turns, this prevents a theoretical analysis of the statistical properties of scaling estimation procedure based on Eq. (1) (see a contrario [22] for one of the only theoretical result). More recently, a new type of multiplicative martingales, referred to as compound Poisson cascades and motions, have been introduced in the literature [5]. They were later generalized to infinitely divisible cascades and motions and fractional Brownian motions in multifractal time [21, 24, 4, 8, 9]. Such processes are regarded as fruitful alternatives to the original and celebrated Mandelbrot's cascades, as they enable to define processes with both known multifractal properties and stationary increments. Moreover, they are very easy to simulate. Therefore, they provide practitioners with relevant and efficient models both for the analysis and the modeling of multifractal properties in real world data. This is why they are studied in details in the present contribution. Yet, to our knowledge, neither the dependence structures of their wavelet or increment coefficients nor the impact on such structures of varying the number of vanishing moments of the analyzing wavelet, or of the order of the increments, have so far been studied.

Multiresolution analysis. To study scale invariance and scaling properties, increments have been the most commonly used multiresolution coefficients and their statistical properties are traditionally and classically analyzed in the literature. However, it is now well known that wavelet coefficients provide relevant, accurate and robust analysis tools for the practical analysis of scale invariance (self-similarity and long range dependence notably, cf. [1]). Furthermore, it has recently been shown [17, 18] that the accurate analysis of multifractal properties requires the use of wavelet Leaders instead of wavelet coefficients.

Goals and contributions. The central aim of the present contribution consists of studying the impact of varying the number of vanishing moments of the analyzing wavelet or the order of the increments on the correlation and dependence (or higher order correlation) structures of three different multiresolution coefficients (increment, wavelet and Leader coefficients) for two different classes of multifractal stochastic processes, infinitely divisible motions and fractional Brownian motions in multifractal time. This is achieved by combining analytical studies and numerical analysis. The major contribution is to show that, while increasing the number of vanishing moments of the analyzing wavelet or the order of the increments significantly fasten the decay of the correlation functions of the increment and wavelet coefficients, in a manner comparable to what is observed for fBm, it does not impact at all the decay of higher order correlation functions.

Outline. The key definitions and properties of infinitely divisible cascades, motions and fractional Brownian motions in multifractal time are recalled in Section 2. The three different types of multiresolution quantities (increment, wavelet and leader coefficients) are defined and related one to the other in Section 3. This section also briefly recalls scaling exponent estimation procedures. Analytical results regarding the correlation and higher order correlation functions of increment and wavelet coefficients are carefully detailed in Section 4. Results and analyses obtained from numerical simulations for the higher order correlation functions of increment, wavelet and Leader coefficients are reported in Section 5. Conclusions on the impact of varying the number of vanishing moments of the analyzing wavelet or the order of the increment on correlation and on higher order correlation function decays are discussed in Section 6. The proofs of all analytical results are postponed to Section 7.

2 Infinitely divisible processes

2.1 Infinitely divisible cascade

Infinitely divisible measure. Let G be an infinitely divisible distribution [11], with moment generating function

$$\widetilde{G}(q) = \exp[-\rho(q)] = \int \exp[qx] dG(x).$$

Let M denote an infinitely divisible, independently scattered random measure distributed by the infinitely divisible distribution G, supported by the time-scale half plane $\mathscr{P}^+ = \mathbb{R} \times \mathbb{R}^+$ and associated to its control measure $dm(t,r)$. This assumption means that the measure of a set A is the random variable $m(A)G$. Since G is infinitely divisible, the Lévy-Khintchine formula applies; it has the following remarkable consequence on the characteristic function of M: For any Borel set $\mathscr{E}$,

$$\mathbb{E}[\exp[qM(\mathscr{E})]] = \exp[-\rho(q)m(\mathscr{E})] = \exp\left[-\rho(q)\int_{\mathscr{E}} dm(t,r)\right]. \tag{3}$$

Infinitely divisible cascade. Let $\mathscr{C}_r(t)$, $0 < r < 1$, denote a so-called truncated influence cone, defined as

$$\mathscr{C}_r(t) = \{(t',r') : r \le r' \le 1, t - r'/2 \le t' < t + r'/2\}.$$

Following the definition of compound Poisson cascades [5] (see paragraph below), it has been proposed in [24, 4, 8, 9] to define infinitely divisible cascades (or noises) as follows.

Definition 1. An Infinitely Divisible Cascade (or Noise) (IDC) is a family of processes $Q_r(t)$, parametrized by $0 < r < 1$, of the form

$$Q_r(t) = \frac{\exp[M(\mathscr{C}_r(t))]}{\mathbb{E}[\exp[M(\mathscr{C}_r(t))]]} = \exp[\rho(1)m(\mathscr{C}_r(t))]\exp[M(\mathscr{C}_r(t))]. \tag{4}$$

Let us moreover define

$$\tilde{\varphi}(q) = \rho(q) - q\rho(1) = -\log\left(\frac{\mathbb{E}[e^{qX}]}{\mathbb{E}[e^X]^q}\right) = -\log\left(\frac{\mathbb{E}[Z^q]}{(\mathbb{E}[Z])^q}\right),$$

whenever defined, with $Z = \exp[X]$ and X distributed according to the infinitely divisible distribution G. Obviously, $\tilde{\varphi}(q)$ is a concave function, with $\tilde{\varphi}(0) = \tilde{\varphi}(1) = 0$. Note that the cone of influence is truncated at the scale r, so that details of smaller scale are absent in the construction of the cascade. The mathematical difficulty lies in understanding the limit when $r \to 0$, in which case all (small) scales will be present.

Control measure. In the remainder of the text, following [4, 8], the control measure is chosen such that $dm(t,r) = \mu(dr)dt$. The choice of the shift-invariant Lebesgue measure dt will have the consequence that the processes constructed below have stationary increments. Following [4], $\mu(dr)$ is set to $\mu(dr) = c(dr/r^2 + \delta_{\{1\}}(dr))$, where $\delta_{\{1\}}(dr)$ denotes a point mass at $r = 1$.

Central property. With this choice of measure, an infinitely divisible cascade possesses the following key property:

Proposition 1. *Let Q_r be an infinitely divisible cascade,* $0 < r < 1$. *Then,* $\forall q > 0$, *such that* $\mathbb{E}[Q_r^q(t)] < \infty$, $\forall t \in \mathbb{R}$,

$$\mathbb{E}[Q_r^q(t)] = \exp[-\tilde{\varphi}(q)m(\mathscr{C}_r(t))] = r^{\varphi(q)}, \tag{5}$$

where, for ease of notation, $\varphi(q) \equiv c\tilde{\varphi}(q)$.

Integral scale. By construction, infinitely divisible cascades Q_r are intrinsically tied to a characteristic scale, referred to as the *integral scale*, in the hydrodynamic turbulence literature (cf. e.g., [20]). Essentially, it results from the fact that the influence cone $\mathscr{C}_r(t)$ is limited above by an upper limit $r \leq r' \leq L$. Traditionally, and without loss of generality, the upper limit is assigned to the reference value $L \equiv 1$, as only the ratio L/r controls the properties of Q_r (cf. [8]). Therefore, in what follows, we study the properties of the different processes defined from Q_r only for $0 \leq t \leq L \equiv 1$.

Compound Poisson cascade. Compound Poisson cascades (CPC) [5] consist of a particular instance within the IDC family. They are obtained by taking for infinitely divisible random measure M a sum of weighted Dirac masses, as follows: One considers a Poisson point process (t_i, r_i) with control measure $dm(t,r)$, and one associates to each of the points (t_i, r_i) positive weights which are i.i.d. random variables W_i. Then, the definition Eq. (4) reads [5]:

$$Q_r(t) = \frac{\exp[\sum_{(t_i,r_i)\in\mathscr{C}_r(t)]} \log W_i}{\mathbb{E}[\exp[\sum_{(t_i,r_i)\in\mathscr{C}_r(t)} \log W_i]]} = \frac{\prod_{(t_i,r_i)\in\mathscr{C}_r(t)} W_i}{\mathbb{E}[\prod_{(t_i,r_i)\in\mathscr{C}_r(t)} W_i]}. \tag{6}$$

When the control measure $dm(t,r)$ is chosen as above, CPC satisfies Eq. (5) with [5]:

$$\varphi(q) = c[(1-\mathbb{E}[W^q]) - q(1-\mathbb{E}W)], \tag{7}$$

whenever defined.

2.2 *Infinitely divisible motion*

The following remarkable result states that the random cascades $Q_r(t)$ have a weak limit when $r \to 0$ (i.e. their distribution functions have a pointwise limit).

Proposition 2. *Let $Q_r(t)$ denote an infinitely divisible cascade and $A_r(t) = \int_0^t Q_r(u)du$ be its probability density function (PDF). There exists a càdlàg (right-continuous with left limits) process $A(.)$ such that almost surely*

$$A(t) = \lim_{r\to 0} A_r(t),$$

for all rational t simultaneously. This process A is a well defined process on condition that $\varphi'(1^-) \geq -1$.

Clearly, the process A is also increasing, and therefore, it is a PDF; it is called an Infinitely Divisible Motion.

Remark 1: Since $Q_r > 0$, all processes A_r and A are non-decreasing and therefore have right and left limits; thus, A can be extended to a càdlàg process defined on $\mathbb{R}$.

Proposition 3. *The process A is characterized by the following properties [5, 4, 8]:*

1. *It possesses a scaling property of the form,* $\forall 0 < q < q_c^+$, $0 < t < 1$*:*

$$\mathbb{E}A(t)^q = C(q)t^{q+\varphi(q)}, \tag{8}$$

   with $C(q) > 0$ *and*

$$q_c^+ = \ sup\ \{q \geq 1, q+\varphi(q)-1 \geq 0\}\ ; \tag{9}$$

2. *Its increments are stationary w.r.t. time t, hence, the covariance structure of A necessarily reads* $\mathbb{E}A(t)A(s) = f(|t|) + f(|s|) - f(|t-s|)$ *;*
3. *Combining the two previous items immediately yields the detailed form of the covariance :*

$$\mathbb{E}A(t)A(s) = \sigma_A^2(|t|^{2+\varphi(2)} + |s|^{2+\varphi(2)} - |t-s|^{2+\varphi(2)}),\ |t-s| \leq 1. \tag{10}$$

   The constant of the covariance can be computed explicitly: $\sigma_A^2 \equiv \mathbb{E}A(1)^2 = ((1+\varphi(2))(2+\varphi(2)))^{-1}$ *[27].*
4. *The multifractal*[1] *properties of the sample paths* $A(t)$ *are entirely controlled by the only function* $q+\varphi(q)$*. This can be inferred from the results in [5].*

2.3 Fractional Brownian motion in multifractal time

The stationary increment infinitely divisible motion $A(t)$ suffers from a severe limitation as far as data modeling is concerned: It is a monotonously increasing process. Following the original idea proposed in [21], it has been proposed to overcome this drawback by subordinating it to fractional Brownian motion:

Definition 2. Let B_H denote fBm with self-similarity parameter $0 < H < 1$ and $A(t)$ a stationary increment infinitely divisible motion $A(t)$. The process

$$\mathscr{B}(t) = B_H(A(t)) \tag{11}$$

is referred to as fBm in multifractal time (MF-fBm, in short).

The properties of $\mathscr{B}(t)$ stem from the combination of those of B_H and of the random time change A that was performed:

Proposition 4.
1. *The process* $\mathscr{B}(t)$ *possesses the following scaling property:*

$$\forall 0 < q < q_c^+/H \quad \mathbb{E}|\mathscr{B}(t)|^q = c_q|t|^{qH+\varphi(qH)}; \tag{12}$$

2. *Its increments are stationary with respect to time t ;*

[1] For a thorough introduction to multifractal analysis, the reader is referred to e.g., [17].

3. *Combining stationary increments and scaling yields the following covariance function:*

$$\mathbb{E}(\mathscr{B}(t)\mathscr{B}(s)) = \sigma_{\mathscr{B}}^2 \left(|t|^{2H+\varphi(2H)} + |s|^{2H+\varphi(2H)} - |t-s|^{2H+\varphi(2H)} \right), \qquad (13)$$

where $\sigma_{\mathscr{B}}^2 = \mathbb{E}|B_H(1)|^2/2$.
4. *The multifractal properties of the sample paths* $\mathscr{B}(t)$ *are entirely controlled by the only function* $qH + \varphi(qH)$.

3 Multiresolution quantities and scaling parameter estimation

3.1 Multiresolution quantities

Increments. The increments of order P, of a function X, taken at lag τ are defined as:

$$X_{(\tau,P)}(t) \equiv \sum_{k=0}^{P} (-1)^k \binom{k}{p} X(t - k\tau). \qquad (14)$$

Wavelet coefficients. The discrete wavelet transform (DWT) coefficients of X are defined as

$$d_X(j,k) = \int_{\mathbb{R}} X(t)\, 2^{-j} \psi_0(2^{-j}t - k)\, dt. \qquad (15)$$

The mother-wavelet $\psi_0(t)$ consists of an oscillating reference pattern, chosen such that the collection $\{2^{-j/2}\psi_0(2^{-j}t-k), j \in \mathbb{N}, k \in \mathbb{N}\}$ forms an orthonormal basis of $L^2(\mathbb{R})$. Also, it is characterized by its *number of vanishing moments*: an integer $N_\psi \geq 1$ such that $\forall k = 0,1,\ldots,N_\psi - 1$, $\int_{\mathbb{R}} t^k \psi_0(t)dt \equiv 0$ and $\int_{\mathbb{R}} t^{N_\psi} \psi_0(t)dt \neq 0$. This N_ψ controls the behavior of the Fourier transform Ψ_0 of ψ_0 at origin:

$$\Psi_0(\nu) \sim |\nu|^{N_\psi},\ |\nu| \to 0. \qquad (16)$$

Wavelet leaders. In the remainder of the text, we further assume that $\psi_0(t)$ has a compact time support. Let us define dyadic intervals as $\lambda = \lambda_{j,k} = \left[k2^j, (k+1)2^j\right)$, and let 3λ denote the union of the interval λ with its 2 adjacent dyadic intervals: $3\lambda_{j,k} = \lambda_{j,k-1} \cup \lambda_{j,k} \cup \lambda_{j,k+1}$. Following [17, 29], wavelet Leaders are defined as:

$$L_X(j,k) = L_\lambda = \sup_{\lambda' \subset 3\lambda} |d_{X,\lambda'}|. \qquad (17)$$

The wavelet Leader $L_X(j,k)$ practically consists of the largest wavelet coefficient $|d_X(j',k')|$ at all finer scales $2^{j'} \leq 2^j$ in a narrow time neighborhood. It has been shown theoretically that the analysis of the multifractal properties of sample paths of stochastic processes and particularly the estimation of their scaling parameters is relevant and accurate only if wavelet Leaders, rather than wavelet coefficients or increments, are chosen as multiresolution quantities (cf. [17, 29]).

Vanishing moments. Increments, as defined in Eq. (14) above, can be read as wavelet coefficients, obtained from the specific mother-wavelet

$$\psi_0^I(t) = \sum_{k=0}^{P} (-1)^k \binom{k}{p} \delta(t-k)$$

(where δ is the Dirac mass function):

$$X_{(2^j,P)}(2^j k) \equiv d_X(j,k;\psi_0^I).$$

It is straightforward to check that the number of vanishing moments of this particular mother-wavelet corresponds to the increment order: $N_\psi \equiv P$. This explains why increments are sometimes referred to as wavelet coefficients, obtained from a low regularity mother-wavelet (historically referred to as the *poor man's wavelet* [14]). Moreover, increments are often regarded as *practical derivation of order P*, the same interpretation holds for wavelet coefficients, computed from a mother wavelet with

$$N_\psi = P \geq 1 \tag{18}$$

vanishing moments. Hence, N_ψ and P play similar roles.

Multiresolution analysis. From the definitions above, it is obvious that varying the lag τ corresponds equivalently to changing the analysis scale $a = 2^j$. Therefore, increments consists of multiresolution quantities, in the same spirit as wavelet coefficients and Leaders do. In the present contribution, we therefore analyze in a common framework the three types of multiresolution quantities, $T_X(a,t)$, increment, wavelet and Leader coefficients:

$$\begin{aligned} T_X(2^j, 2^j k) &= X_{2^j,P}(2^j k) \\ &= d_X(j,k), \\ &= L_X(j,k). \end{aligned}$$

3.2 Scaling parameter estimation procedures

Inspired from Eq. (1), classical scaling parameter $\zeta(q)$ estimation procedures are based on linear regressions, over dyadic scales $a_j = 2^{j_1}, \ldots, 2^{j_2}$ ($\sum$ stands for $\sum_{j=j_1}^{j_2}$, the weights w_j satisfy $\sum w_j = 0$ and $\sum j w_j = 1$) (cf. [19, 28] for details):

$$\widehat{\zeta}(q) = \sum w_j \log_2 \left(\frac{1}{n_j} \sum_{k=1}^{n_j} |T_X(2^j, 2^j k)|^q \right). \tag{19}$$

The analysis of the statistical performance of such estimation procedures requires the knowledge of the multivariate dependence structures of the variables $|T_X(a,t)|^q$. Some aspects of these dependence structure are studied in the next sections.

4 Dependence structures of the multiresolution coefficients: analytical study

The aim of the present section is to study analytically, both for infinitely divisible motion and for fractional Brownian motion in multifractal time, $X = A, \mathscr{B}$, the impact of varying P or N_ψ on some aspects of the dependence structures of the $T_X(a,t)$, when the $T_X(a,t)$ are either increments or discrete wavelet coefficients.

Section 4.1 starts with the theoretical analyses of the covariance functions $\mathbb{E}T_X(a,t)T_X(a,s)$.

Section 4.2 continues with the theoretical studies of the *higher order* covariance functions $\mathbb{E}|T_X(a,t)|^q|T_X(a,s)|^q$, for some qs: The key point being that the absolute values, $|\cdot|$, consisting of non linear transforms of the $T_X(a,t)$ involve the whole dependence structure.

All results reported in Sections 4 and 5 are stated for $0 < t,s < 1$, hence for $|t-s| < 1$ (i.e., within the integral scale).

4.1 Correlation structures for increment and wavelet coefficients

4.1.1 Increments

From Eq. (10) (Eq. (13), resp.), it can be shown that the covariance structure of the increments of order P of Infinitely divisible motion A (Fractional Brownian motion in multifractal time $\mathscr{B}$, resp.) reads:

$$\mathbb{E}A_{(\tau,P)}(t)A_{(\tau,P)}(s) = \sigma_A^2 \sum_{n=0}^{P} (-1)^n \binom{n}{P} (t-s+(n-P)\tau)^{2+\varphi(2)}, \tag{20}$$

$$\mathbb{E}\mathscr{B}_{(\tau,P)}(t)\mathscr{B}_{(\tau,P)}(s) = \sigma_{\mathscr{B}}^2 \sum_{n=0}^{P} (-1)^n \binom{n}{P} (t-s+(n-P)\tau)^{2H+\varphi(2H)}. \tag{21}$$

Taking the limit $\tau \to 0$, i.e., $1 > |t-s| \gg \tau$ yields:

$$\lim_{\tau\to 0} \frac{\mathbb{E}A_{(\tau,P)}(t)A_{(\tau,P)}(s)}{|\tau|^{2P}} = C_P(\varphi(2))|t-s|^{2+\varphi(2)-2P}, \tag{22}$$

$$\lim_{\tau\to 0} \frac{\mathbb{E}\mathscr{B}_{(\tau,P)}(t)\mathscr{B}_{(\tau,P)}(s)}{|\tau|^{2P}} = C_{P,H}(\varphi(2H))|t-s|^{2H+\varphi(2H)-2P}. \tag{23}$$

The direct proofs are not given here, they are detailed in [27]. Such results can also be obtained by following the proof of Proposition 5 below.

4.1.2 Wavelet coefficients

Let ψ_0 denote a compact support mother wavelet with N_ψ vanishing moments.

Proposition 5. *Let $j,k,k' \in \mathbb{Z}$, $|k-k'| \leq C2^{-j}$ such that the supports of $\psi_{j,k}$ and $\psi_{j,k'}$ are included in $[0,1]$. Then*

$$\lim_{j\to-\infty} \frac{\mathbb{E} d_A(j,k) d_A(j,k')}{2^{2jN_\psi}} \sim O(|2^j k - 2^j k'|)^{(2+\varphi(2)-2N_\psi)}, \tag{24}$$

$$\lim_{j\to-\infty} \frac{\mathbb{E} d_{\mathscr{B}}(j,k) d_{\mathscr{B}}(j,k')}{2^{2jN_\psi}} \sim O(|2^j k - 2^j k'|)^{(2H+\varphi(2H)-2N_\psi)}. \tag{25}$$

Proof. We follow step by step the calculations conducted in [13] on fractional Brownian motion (note that we use a L^1-normalization for the wavelet coefficients instead of the usual L^2-normalization, as in [13]).

From the form of the covariance function (cf. Eq. (10)), one obtains:

$$\mathbb{E} d_A(j,k) d_A(j,k') = -\sigma_A^2 (2^j)^{2+\varphi(2)} \int \Big(R_\psi(1, \tau - (k-k')) |\tau|^{1+\varphi(2)/2} \Big) d\tau$$

with

$$R_\psi(\alpha,\tau) = \sqrt{\alpha} \int_{-\infty}^{+\infty} \psi_0(t) \psi_0(\alpha t - \tau) dt$$

the reproducing kernel of ψ_0. Rewriting the relation above in the Fourier domain and using Eq. (16) yields:

$$\mathbb{E} d_A(j,k) d_A(j,k') = \sigma_{A,\psi_0}^2 (2^j)^{2+\varphi(2)+1} \int_{-\infty}^{+\infty} e^{i\omega(k-k')} \frac{|\Psi_0(\omega)|^2}{|\omega|^{2+\varphi(2)+1}} \frac{d\omega}{2\pi},$$

with $\sigma_{A,\psi_0}^2 = \sigma_A^2 2 \sin(\frac{\pi}{2}(2+\varphi(2))) \Gamma(4+2\varphi(2)+1)$, and hence:

$$\mathbb{E} d_A(j,k) d_A(j,k') = 2^{j(2+\varphi(2)} O(|k-k'|^{2+\varphi(2)-2N_\psi}).$$

Now, since $0 \leq |k-k'| \leq C2^{-j}$ (which means that the support of $\psi_{j,k}$ is centered on a point $t = c2^j k$), we can write $k = 2^{-j} p$, $k' = 2^{-j} p'$ for $|p-p'| \leq C$, and

$$\mathbb{E} d_A(j,k) d_A(j,k') = \mathbb{E} d_A(j,2^{-j}p) d_A(j,2^{-j}p') = 2^{2jN_\psi} O(|p-p'|^{2+\varphi(2)-2N_\psi}).$$

4.1.3 Vanishing moments and correlation

These computations show two striking results regarding the impact of varying N_ψ (resp., P) on the covariance function of the wavelet coefficients (resp., the increments). First, the order of the leading term in $|\tau$ of the asymptotic expansion, $|\tau| \to 0$, increases with N_ψ (resp., P). Second, the decrease in $|t-s|$ of the coefficient of the leading term is faster when N_ψ (resp., P) increases. To conduct comparisons against results obtained for fractional Brownian motions, this shows first that the limit $|t-s| \to +\infty$ needs to be replaced with the limit $|\tau| \to 0$ and the range $\tau \le lt-s| \le 1$. Then, one observes that the impact of varying N_ψ (resp., P) on the correlation structures of the wavelet coefficients (resp., the increments) of Infinitely Divisible Motion and fractional Brownian motion in multifractal time is equivalent, mutatis mutandis, to that obtained for fractional Brownian motion: The larger N_ψ (resp., P), the faster the decay of the correlation functions.

Such results call for the following comments:

- This comes as no surprise as both processes A and $\mathscr{B}$ share with B_H two key properties: scale invariance and stationary increments.
- Because A and $\mathscr{B}$ are non Gaussian processes, the derivation of their correlation structures does not induce the knowledge of their dependence (or higher order correlation) structures. This is why functions $\mathbb{E}|T_X(a,t)|^q|T_X(a,s)|^q$ are further analyzed in the next sections.
- No analytical results are available for the correlation of the wavelet Leaders, $\mathbb{E}L_X(j,k)L_X(j,k')$. This is because while increment and wavelet coefficients are obtained from linear transforms of X, the Leaders $L_X(j,k)$ consists of non linear transforms (as do the $|T_X(a,t)|^q$ in general).
- Because $\varphi(1) \equiv 0$, the key quantity $\varphi(2)$ can be rewritten $\varphi(2) - 2\varphi(1)$.

4.2 Higher order correlations for increments

4.2.1 First order increments

Infinitely divisible motion A**.** The covariance function for the integer q-th power of the first order increments of A can be obtained analytically:

Theorem 1. *Let* $1 \le q < q_c^+/2$ *be an integer. There exists* $c(q) > 0$ *such that, for* $0 < t,s < 1$*,*

$$\lim_{\tau \to 0} \frac{\mathbb{E}A_{(\tau,1)}{}^q(t)A_{(\tau,1)}{}^q(s)}{c(q)|\tau|^{2(q+\varphi(q))}} = |t-s|^{\varphi(2q)-2\varphi(q)}.$$

The constant $c(q)$ *can be calculated precisely from* $c(q)|\tau|^{2(q+\varphi(q))} = (\mathbb{E}A_{(\tau,1)}{}^q(1))^2$.

The proof of Theorem 1 is postponed to Section 7.2.

For non integer $q \notin \mathbb{N}$, an exact result for scaling is not available, yet the following inequalities can be obtained, that show that the exact power behavior obtained for integer q extend to real q, at least in the limit $|t-s| \to 0$.

Theorem 2. *Let* $1 \le q < q_c^+/2$*. There exists* $C_1 > 0$ *and* $C_2 > 0$ *depending only on* q *such that,*

$$C_1|t-s|^{\varphi(2q)-2\varphi(q)} \leq \lim_{\tau\to 0} \frac{\mathbb{E}A_{(\tau,1)}{}^q(t)A_{(\tau,1)}{}^q(s)}{|\tau|^{2(q+\varphi(q))}} \leq C_2|t-s|^{\varphi(2q)-2\varphi(q)}.$$

In particular,

$$\begin{aligned}\varphi(2q)-2\varphi(q) &= \inf\{\alpha \in \mathbb{R};\ \lim_{|t-s|\to 0} \frac{1}{|t-s|^{\alpha}} \lim_{\tau\to 0} \frac{\mathbb{E}A_{(\tau,1)}{}^q(t)A_{(\tau,1)}{}^q(s)}{|\tau|^{2(q+\varphi(q))}} = +\infty\} \\ &= \sup\{\alpha \in \mathbb{R};\ \lim_{|t-s|\to 0} \frac{1}{|t-s|^{\alpha}} \lim_{\tau\to 0} \frac{\mathbb{E}A_{(\tau,1)}{}^q(t)A_{(\tau,1)}{}^q(s)}{|\tau|^{2(q+\varphi(q))}} = 0\}.\end{aligned}$$

This is proven in Section 7.3.

Moreover, it is also of interest to consider the dependence of increments taken at two different analyzing scales, τ_1 and τ_2:

Corollary 1. *(Of the proof of theorem 7.2) Let $1 \leq q < q_c^+/2$ be an integer. There exists $c(q) > 0$ such that,*

$$\lim_{\tau_1,\tau_2\to 0} \frac{\mathbb{E}A^q_{(\tau_1,1)}(t)A^q_{(\tau_2,1)}(s)}{c(q)|\tau_1\tau_2|^{q+\varphi(q)}} = |t-s|^{\varphi(2q)-2\varphi(q)}.$$

This shows that the power law behavior in Theorem 7.3 for increments taken at the same scale τ can be extended to increments defined at any two different scales.

Fractional Brownian motion in multifractal time $\mathscr{B}$

Proposition 6. *Let $1 > H > 1/2$. There exists two constants $C_1 > 0$ and $C_2 > 0$ such that,*

$$C_1|t-s|^{\varphi(4H)-2\varphi(2H)} \leq \lim_{\tau\to 0} \frac{\mathbb{E}\mathscr{B}^2_{(\tau,1)}(t)\mathscr{B}^2_{(\tau,1)}(s)}{\tau^{4H+2\varphi(2H)}} \leq C_2|t-s|^{\varphi(4H)-2\varphi(2H)}.$$

The proof, which mostly relies on the use of Theorem 2, with $q = 2H$, is postponed to Section 7.4.

4.2.2 Second order increments

Let us now study the correlation of the squared ($q = 2$) second order ($P = 2$) increments of Infinitely divisible motion A and fractional Brownian motion in multifractal time $\mathscr{B}$.

Proposition 7. *Let φ be chosen such that $q_c > 4$ (cf. Eq. (9)). There exists $c > 0$ such that,*

$$\lim_{\tau\to 0} \frac{\mathbb{E}A_{(\tau,2)}{}^2(t)A_{(\tau,2)}{}^2(s)}{\mathbb{E}A_{(\tau,2)}{}^2(t)\mathbb{E}A_{(\tau,2)}{}^2(s)} = \lim_{\tau\to 0} \frac{\mathbb{E}A_{(\tau,2)}{}^2(t)A_{(\tau,2)}{}^2(s)}{c|\tau|^{2(\varphi(2)+2)}} = |t-s|^{\varphi(4)-2\varphi(2)}.$$

The proof is detailed in Section 7.5.

Proposition 8. *Let φ be chosen such that $q_c > 4$ (cf. Eq. (9)) and let $1 > H > 1/2$. There exists two constants $C_1 > 0$ and $C_2 > 0$ such that, for all $0 \leq t < s \leq 1$,*

$$C_1 |t-s|^{\varphi(4H)-2\varphi(2H)} \leq \lim_{\tau \to 0} \frac{\mathbb{E}\mathscr{B}^2_{(\tau,2)}(t)\mathscr{B}^2_{(\tau,2)}(s)}{\tau^{4H+2\varphi(2H)}} \leq C_2 |t-s|^{\varphi(4H)-2\varphi(2H)}.$$

The proof is detailed in Section 7.6. This result suggests that the higher order covariance functions of $\mathscr{B}$ have the same behaviors as those of A, replacing q with qH.

4.3 Role of the order of the increments

Comparing, on one hand, the results of Theorem 1 (for $\mathbb{E}A^2_{\tau,1}(t)A^2_{\tau,1}(s) \equiv \mathbb{E}|A_{\tau,1}(t)|^2|A_{\tau,1}(s)|^2$) versus those of Proposition 7 (for $\mathbb{E}A^2_{\tau,2}(t)A^2_{\tau,2}(s) \equiv \mathbb{E}|A_{\tau,2}(t)|^2|A_{\tau,2}(s)|^2$) and, on other hand, the results of Proposition 6 (for $\mathbb{E}\mathscr{B}^2_{\tau,1}(t)\mathscr{B}^2_{\tau,1}(s) \equiv \mathbb{E}|\mathscr{B}_{\tau,1}(t)|^2|B_{\tau,1}(s)|^2$) versus those of Proposition 8 (for $\mathbb{E}B^2_{\tau,2}(t)B^2_{\tau,2}(s) \equiv \mathbb{E}|B_{\tau,2}(t)|^2|B_{\tau,2}(s)|^2$), yields the first major conclusion: Increasing P from 1 to 2 induces, for the higher order correlation functions, neither a change in the order in $|\tau|$ of the leading term of the asymptotic expansion in $|\tau| \to 0$, nor any faster decay of the coefficient in $|t-s|$ for this leading term. This is in clear contrast with the impact of P on the correlation functions $\mathbb{E}A_{\tau,P}(t)A_{\tau,P}(s)$ and $\mathbb{E}\mathscr{B}_{\tau,P}(t)\mathscr{B}_{\tau,P}(s)$, with $P = 1,2$ (cf. results of Section 4.1).

5 Dependence structures of the multiresolution coefficients: Conjectures and numerical studies

5.1 Conjectures

The analytical results obtained in Section 4 for the q-th ($q = 1,2$) power of the absolute values of the first and second order increments of both processes $X = A, \mathscr{B}$ lead us to formulate the two following conjectures, for the three different multiresolution quantities $T_X(a,t)$ considered here (increment, wavelet and Leader coefficients).

Conjecture 1. Let $1 \leq q < q_c^+/2$. There exists $C_A(q) > 0$ depending only on q such that, for $0 < s-t < 1$, one has

$$\lim_{a \to 0} \frac{\mathbb{E}|T_A(a,t)|^q |T_A(a,s)|^q}{|a|^{2(q+\varphi(q))}} = C_A(q)|t-s|^{\varphi(2q)-2\varphi(q)}.$$

Conjecture 2. Let $1 \leq qH < q_c^+/2$. There exists $C_{\mathscr{B}}(q) > 0$ depending only on q such that, for $0 < s-t < 1$, one has

$$\lim_{a\to 0} \frac{\mathbb{E}|T_{\mathscr{B}}(a,t)|^q |T_{\mathscr{B}}(a,s)|^q}{|a|^{2(qH+\varphi(qH))}} = C_{\mathscr{B}}(q)|t-s|^{\varphi(2qH)-2\varphi(qH)}.$$

The central features of these two conjectures consists of the following facts: i) the higher order covariance functions decay algebraically (by concavity of the function $\varphi(q)$, for all $q > 0$, the quantity $\varphi(2q) - 2\varphi(q)$ is strictly negative) ; ii) the scaling exponent of the leading term characterizing the algebraic decay in $|t-s|$ is not modified when the order P of the increments or the number of vanishing moment N_ψ of the mother wavelet are increased.

5.2 *Numerical simulations*

To give substance to these conjectures, formulated after the analytical results obtained in Section 4, the following sets of numerical simulations are performed and analyzed.

5.2.1 Simulation set up

Numerical simulations are conducted on compound Poisson motions (i.e., processes A obtained from compound Poisson cascades) rather than on infinitely divisible motions, as the former are much easier to handle from a practical synthesis perspective. The practical synthesis of realizations of the latter is linked with heavy computational and memory costs, which impose severe practical limitations (in maximally possible sample size, for instance). Therefore, they remain barely used in applications. In contrast, realizations of processes A and $\mathscr{B}$ based on compound Poisson cascades are reasonably easy to simulate numerically (cf. [8] for a review). For ease of notations, the corresponding processes A and $\mathscr{B}$ are referred to as CPM and CPM-MF-fBm.

More specifically, we illustrate results with log-Normal multipliers $W = \exp[V]$, where $V \overset{i.i.d.}{\sim} \mathscr{N}(\mu, \sigma^2)$ are Gaussian random variables. In this case, the function $\varphi(q)$ Eq. (7) is given by:

$$\varphi(q) = c\left[\left(1-\exp\left(\mu q + \frac{\sigma^2}{2}q^2\right)\right) - q\left(1-\exp\left(\mu + \frac{\sigma^2}{2}\right)\right)\right]. \qquad (26)$$

This choice is motivated by the fact that it is often considered in applications [7].

For CPM, numerical results reported here are computed with parameters $c = 1$, $\mu = -0.3$ and $\sigma^2 = 0.04$ (hence $q_c^+/2 \approx 8.92$) ; $nbreal = 100$ realizations of the process are numerically synthetised, with sample size $N = 2^{21}$. For CPM-MF-fBm, the same parameters are used, with in addition $H = 0.7$ (hence $(q_c^+/H)2 \simeq 12.74$) and the sample size is reduced to $N = 2^{18}$ (as the synthesis of realizations of CPM-MF-fBm is slightly more involved than for CPM, hence limiting the obtainable sample size to a smaller one, as compared to CPM).

From these, the increment, wavelet and Leader coefficients of CPM A and CPM-MF-fBm $\mathscr{B}$ are calculated for a chosen analysis scale a. Results are illustrated here for $a = 2^3$, but similar conclusions are drawn for any different a. The number of vanishing moments (and increment orders) are set to $P \equiv N_\psi = \{1, 2, 5, 8\}$. The statistical orders q are in the range $0 < q < q_c/2$.

Both the synthesis and analysis codes are developed by ourselves in MATLAB and are available upon request.

5.2.2 Goal and analysis

Following the formulation of Conjectures 1 and 2 above, the goal of the simulations is to validate the power law decay of the correlation functions of the q-th power of the absolute values of the multiresolution quantities and to estimate the corresponding power law exponents $\alpha_A(q, N_\psi) = \varphi(2q) - 2\varphi(q)$ and $\alpha_{\mathscr{B}}(q, N_\psi) = \varphi(2qH) - 2\varphi(qH)$, controlling respectively such decays.

To do so, we make use of the wavelet based spectral estimation procedure documented and validated in [26], whose key features are briefly recalled here. Let Y denote a second order stationary process with covariance function, $\mathbb{E}Y(t)Y(s) \simeq |t - s|^{-\alpha}$. Then, it has been shown that $1/n_j \sum_k^{n_j} d_Y^2(j,k) \simeq 2^{j\alpha}$. Therefore, the parameter α can be estimated from a linear regression in the diagram $\log_2 1/n_j \sum_k^{n_j} d_X^2(j,k)$ vs. $\log_2 2^j = j$:

$$\hat{\alpha} = \sum_{j=j_1}^{j_2} w_j \log_2 \left(1/n_j \sum_k^{n_j} d_Y^2(j,k) \right),$$

the weights w_j satisfy $\sum_{j=j_1}^{j_2} w_j = 0$ and $\sum_{j=j_1}^{j_2} j w_j = 1$. This wavelet based estimation procedure is applied to time series consisting of the q-th power of the absolute value of increment, wavelet and Leader coefficients of A and $\mathscr{B}$ computed at an arbitrary scale a: $Y(t) \equiv |T_X(a,t)|^q$. Estimation is performed with a compact support Daubechies wavelet with 4 vanishing moments.

5.2.3 Results and analyses

For increments, wavelet coefficients and Leaders, for all $0 < q < q_c^+$, for all $N_\psi \equiv P$, for both CPM A and CPM-MF-fBm $\mathscr{B}$, satisfactory power laws behaviors can be observed in diagrams $\log_2 1/n_j \sum_k^{n_j} d_X^2(j,k)$ vs. $\log_2 2^j = j$ and power scaling exponent can be estimated. These plots are not shown here.

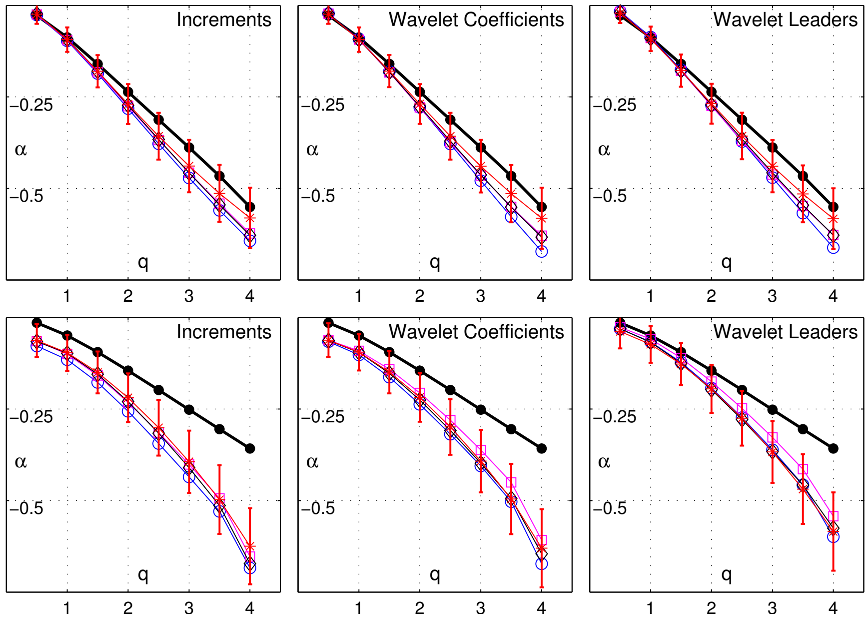

Fig. 1 Wavelet based estimations of power law exponents for the higher correlation functions. Wavelet based estimates (averaged over realizations) of the power law exponents $\hat{\alpha}_A(q,N_\psi)$ (for CPM A, top row) and $\hat{\alpha}_{\mathscr{B}}(q,N_\psi)$ (CPM-MF-fBm $\mathscr{B}$, bottom row) of the higher order correlations of increments (left), wavelet coefficients (center) and wavelet Leaders (right), as functions of q. The conjectured exponents, $\alpha_A(q,N_\psi) = \varphi(2q) - 2\varphi(q)$ and $\alpha_{\mathscr{B}}(q,N_\psi) = \varphi(2qH) - 2\varphi(qH)$, are drawn in fat solid black with dots. The symbols { '$*$' , '$\diamond$' , '$\square$' , '$\circ$' } correspond to $P \equiv N_\psi = \{1,2,5,8\}$, respectively. The error bars (in red) for $N_\psi = P = 1$ correspond to 95% asymptotic confidence intervals.

Fig. 1 represents (in bold black solid with dots) the conjectured power law exponents $\alpha_A(q,N_\psi) = \varphi(2q) - 2\varphi(q)$ (for CPM A, top row) and $\alpha_{\mathscr{B}}(q,N_\psi) = \varphi(2qH) - 2\varphi(qH)$ (CPM-MF-fBm $\mathscr{B}$, bottom row), as a function of q, and compares them to averages over realizations of estimates of power law exponents $\hat{\alpha}_A(q,N_\psi)$ and $\hat{\alpha}_{\mathscr{B}}(q,N_\psi)$, for increments (left), wavelet coefficients (center) and wavelet Leaders (right) with $P \equiv N_\psi = \{1,2,5,8\}$ vanishing moments. Also, for $N_\psi = 1$, asymptotic 95% error bars are shown. Error bars for other values of N_ψ are of similar size and omitted for better readability.

The results in Fig. 1 clearly indicate that, both for A and $\mathscr{B}$:

- No significant difference can be observed between the power law exponents estimated from increments, wavelet coefficients, and wavelet Leaders: The differences between the different estimated scaling exponents are very small, and within confidence intervals.

- These power law exponents, characterizing the decay of the higher order correlation functions of the multiresolution quantities, practically do not vary when the number of vanishing moments (or the order of the increments) is increased.

- For A, for fixed q, notably for small q, the differences between the estimated and conjectured scaling exponents remain small, and within confidence intervals. The slight discrepancies for large q values can be interpreted by the fact that only leading order terms are taken into account by Conjecture 1, whereas in practice, for finite size realizations, higher order terms may also contribute (cf. Remark 3 in Section 7.5). For $\mathscr{B}$, estimated scaling exponents appear farther from (and smaller than) the conjectured ones. This discrepancy can be explained by the limited sample size for realizations of CPM-MF-fBm (roughly one order of magnitude smaller than those obtainable for A). Therefore, higher order terms are likely to impact more significantly than for A on the asymptotic expansion of $\mathbb{E}|T_{\mathscr{B}}(a,t)|^q|T_{\mathscr{B}}(a,s)|^q$, and hence to impair the accurate estimation of the power law exponent of the leading term.

- Conjectures are also actually valid for the range $q \in [0,1]$.

These results and analyses strongly support Conjectures 1 and 2. Similar results and equivalent conclusions are drawn from other choices of $\varphi(q)$ and from numerical simulations performed directly on infinitely divisible motions (though of lower sample size, and not reported here).

6 Discussions and conclusions on the role of the number of vanishing moments:

The theoretical studies (reported in Section 4), together with the numerical analyses (conducted in Section 5) yield the following major conclusions regarding the impact of varying N_ψ or P on the higher order correlations functions of $X_{(\tau,P)}(t)$, $d_X(j,k)$ and $L_X(j,k)$.

- The correlation functions of the increments and wavelet coefficients of A and $\mathscr{B}$ decay faster when $N_\psi \equiv P$ is increased. This effect is equivalent to that obtained for fractional Brownian motion, and results from the facts that B_H, A and $\mathscr{B}$ posses a scale invariance property (as in Eqs. (8) or (12)) and stationary increments and wavelet coefficients(and also from the fact that increments and wavelet coefficients consist of linear transform of the processes).

- For the q-th power ($0 < q < q_c^+/2$) of Leaders or of the absolute values of the increments or wavelet coefficients, which consist of non linear transforms of processes A and $\mathscr{B}$, N_ψ and P no longer impact the decay of the correlation functions.

- The power law exponents characterizing the algebraic decay of these correlation functions are found to be identical for increment, wavelet and Leader coefficients.

- These power law exponents are conjectured to be controlled by the only function $\varphi(q)$ underlying the construction of the infinitely divisible cascade for A, and, in addition, parameter H for $\mathscr{B}$: $\alpha_A(q,N_\psi) = \varphi(2q) - 2\varphi(q)$ and $\alpha_{\mathscr{B}}(q,N_\psi) = \varphi(2qH) - 2\varphi(qH)$.

- Furthermore, both estimated and predicted power law exponents are larger than -1. This is significantly so for small q values. This would be the case for most, if not any, choices of function $\varphi(q)$ commonly used in applications. This reveals very slow (algebraic) decay of the covariance functions, a important characteristic with respect to parameter estimation issues.

After the seminal works on self similar fractional Brownian motions and wavelet coefficients [12, 13] or increments [16], we believe that the present theoretical analyses and numerical results are closing a loop: They shed new lights and significantly renew the understanding of the role of the order of the increments and of the number of vanishing moments of the mother wavelet, for wavelet coefficients and Leaders, with respect to the analysis of scale invariance as modeled by infinitely divisible multifractal processes.

7 Proofs

7.1 Key lemma

The proofs of the results obtained in the present contribution relies on the use of the key Lemma in [4], restated here. Let $\varphi(\cdot) = \psi(-i\cdot)$ and let ω_r be defined by $Q_r = e^{\omega_r}$. For $t, t' \geq 0$, one defines:

$$\mathscr{C}_r(t,t') = \mathscr{C}_r(t) \cap \mathscr{C}_r(t')$$

Lemma 1. *Let $q \in \mathbb{N}^*$, $\overrightarrow{t_q} = (t_1, t_2, ..., t_q)$ with $t_1 \leq t_2 \leq ... \leq t_q$ and $\overrightarrow{p_q} = (p_1, p_2, ..., p_q)$. The characteristic function of the vector $\{w_r(t_m)\}_{1\leq m\leq q}$ reads:*

$$\mathbb{E}\left(e^{\sum_{m=1}^q ip_m M(C_r(t_m))}\right) = e^{\sum_{j=1}^q \sum_{k=1}^j \alpha(j,k)\rho_r(t_k - t_j)}$$

where M is the infinitely divisible, independently scattered random measure used in the construction of Q_r,

$$\rho_r(t) = m(\mathscr{C}_r(0,t)),$$

$$\alpha(j,k) = \psi(r_{k,j}) + \psi(r_{k+1,j-1}) - \psi(r_{k,j-1}) - \psi(r_{k+1,j})$$

and

$$r_{k,j} = \begin{cases} \sum_{m=k}^j p_m, & \text{for } k \leq j, \\ 0 & \text{for } k > j. \end{cases}$$

Moreover,

$$\sum_{j=1}^q \sum_{k=1}^j \alpha(j,k) = \psi\left(\sum_{k=1}^q p_k\right).$$

This can be rewritten as

$$\mathbb{E}Q_r^{p_1}(t_1)Q_r^{p_2}(t_2)...Q_m^{p_m}(t_m) = e^{\sum_{j=1}^{q}\sum_{k=1}^{j}\beta(j,k)\rho_r(t_k-t_j)}$$

with

$$\beta(j,k) = \varphi(r_{k,j}) + \varphi(r_{k+1,j-1}) - \varphi(r_{k,j-1}) - \varphi(r_{k+1,j})$$

and

$$\sum_{j=1}^{q}\sum_{k=1}^{j}\beta(j,k) = \varphi\left(\sum_{k=1}^{q} p_k\right).$$

7.2 Proof of Theorem 1

Let us assume that $s > t$ and $s - t > \tau$. Because $\mathbb{E}A_{(\tau,1)}{}^{2q}(x) < \infty$ for all $0 \leq x \leq 1$, one obtains that $\mathbb{E}A_{(\tau,1)}(t)^q A_{(\tau,1)}(s)^q < \infty$ and, using the monotone convergence theorem for the 4th equality,

$$\begin{aligned}
\mathbb{E}A_{(\tau,1)}{}^q(t)A_{(\tau,1)}{}^q(s) &= \mathbb{E}\left(\lim_{r_1\to 0}\int_t^{t+\tau} Q_{r_1}(x)dx\right)^q \left(\lim_{r_2\to 0}\int_s^{s+\tau} Q_{r_2}(y)dy\right)^q \\
&= \mathbb{E}\prod_{i=1}^{q}\lim_{r_{1,i}\to 0}\lim_{r_{2,i}\to 0}\int_t^{t+\tau} Q_{r_{1,i}}(x_i)dx_i \int_s^{s+\tau} Q_{r_{2,i}}(y_j)dy_j \\
&= \mathbb{E}\lim_{r\to 0}\prod_{i=1}^{q}\int_t^{t+\tau} Q_r(x_i)dx_i \int_s^{s+\tau} Q_r(y_i)dy_i \\
&= \lim_{r\to 0}\mathbb{E}\prod_{i=1}^{q}\int_t^{t+\tau} Q_r(x_i)dx_i \int_s^{s+\tau} Q_r(y_i)dy_i \\
&= \lim_{r\to 0}\int_{[t,t+\tau]^q}\int_{[s,s+\tau]^q}\mathbb{E}\prod_{i=1}^{q} Q_r(x_i)Q_r(y_i)d(x_1,...,x_q)d(y_1,...,y_q)
\end{aligned}$$

By symmetry, this yields:

$$\mathbb{E}A_{(\tau,1)}{}^q(t)A_{(\tau,1)}{}^q(s) = (q!)^2\lim_{r\to 0}\int_{D_1}\int_{D_2}\mathbb{E}\prod_{i=1}^{q} Q_r(x_i)Q_r(y_i)d(x_1,...,x_q)d(y_1,...,y_q).$$

where $D_1 = \{t \leq x_1 \leq x_2, ... \leq x_q \leq t+\tau\}$ and $D_2 = \{s \leq y_1 \leq y_2 \leq ... \leq y_q \leq s+\tau\}$.

Let us fix $r < s - t - \tau$ and define $\Delta_r^+ = \{(t,z) \in \mathbb{R}^2; z > r\}$. Using Lemma 1, we can write $I = \mathbb{E}\prod_{i=1}^{q} Q_r(x_i)Q_r(y_i)$ for $t \leq x_1 \leq x_2, ... \leq x_q \leq t+\tau$ and $s \leq y_1 \leq y_2 \leq ... \leq y_q \leq s+\tau$ as the product of 3 terms:

$$I = e^{\sum_{j=1}^{q}\sum_{k=1}^{j}\beta(j,k)\rho_r(x_k-x_j)} e^{\sum_{j=1}^{q}\sum_{k=1}^{j}\beta(j+q,k+q)\rho_r(y_k,y_j)} e^{\sum_{j=1}^{q}\sum_{k=1}^{q}\beta(j,k+q)\rho_r(y_k,x_j)}. \tag{27}$$

The second term (containing $\rho_r(y_k, x_j,))$ controls the behavior in $|t-s|^{\varphi(2q)-2\varphi(q)}$ while the first and third terms, which do not depend on $|t-s|$, yield the multiplicative factor $|\tau|^{2(q+\varphi(q))}$. Indeed, let us consider

$$J = e^{\sum_{j=1}^{q}\sum_{k=1}^{q}\beta(j+q,k)\rho_r(y_j,x_k)}$$

Since $r < s-t-\tau$, $\rho_r(y_k,x_j) = -\ln|y_k - x_j|$ with $s-t-\tau \leq |y_k - x_j| \leq s-t+\tau$, we obtain,

$$\sum_{j=1}^{q}\sum_{k=1}^{q}\beta(j+q,k) = \sum_{j=1}^{2q}\sum_{k=1}^{j}\beta(j,k) - \sum_{j=1}^{q}\sum_{k=1}^{j}\beta(j,k) - \sum_{j=1}^{q}\sum_{k=1}^{j}\beta(j+q,k+q).$$

But, here, with the notations of Lemma 1, $\overrightarrow{p_q} = (1,1,...,1)$, $r_{k,j} = j-k+1$ and $\beta(j,k)$ depends only on $k-j$. Finally, we obtain

$$\sum_{j=1}^{q}\sum_{k=1}^{q}\beta(j+q,k) = \varphi(2q) - 2\varphi(q)$$

and

$$|s-t+\tau|^{\varphi(2q)-2\varphi(q)} \leq J \leq |s-t-\tau|^{\varphi(2q)-2\varphi(q)}.$$

Hence,

$$|s-t+\tau|^{\varphi(2q)-2\varphi(q)} \lim_{r\to 0} L(r,\tau) \leq \mathbb{E}A_{(\tau,1)}{}^{q}(t)A_{(\tau,1)}{}^{q}(s) \leq |s-t-\tau|^{\varphi(2q)-2\varphi(q)} \lim_{r\to 0} L(r,\tau)$$

with

$$L(r,\tau) = (q!)^2 \int_{D_1}\int_{D_2} e^{\sum_{j=1}^{q}\sum_{k=1}^{j}\beta(j,k)\rho_r(x_k-x_j)} e^{\sum_{j=1}^{q}\sum_{k=1}^{j}\beta(j+q,k+q)\rho_r(y_k,y_j)}.$$

Lemma 2.

$$\lim_{r\to 0} L(r,\tau) = (\mathbb{E}A_{(\tau,1)}{}^{q}(t))^2 = c|\tau|^{2(q+\varphi(q))}.$$

PROOF. It is known that $\mathbb{E}A_{(\tau,1)}{}^{q}(t) = c|\tau|^{q+\varphi(q)}$. However, we also have:

$$\begin{aligned}\mathbb{E}A_{(\tau,1)}{}^{q}(t) &= \lim_{r\to 0}\int_{t\leq x_1,\ldots,x_q} \mathbb{E}\prod_{i=1}^{q} Q_r(x_i)d(x_1,\ldots,x_q)\\ &= (q!)\lim_{r\to 0}\int_{D_1} \mathbb{E}\prod_{i=1}^{q} Q_r(x_i)d(x_1,\ldots,x_q)\end{aligned}$$

which yields $\mathbb{E}A_{(\tau,1)}{}^{q}(t)\mathbb{E}A_{(\tau,1)}{}^{q}(s) = \lim_{r\to 0} L(r,\tau)$. Replacing $\lim_{r\to 0} L(r,\tau)$ with $c|\tau|^{2(q+\varphi(q))}$, we obtain

$$|s-t+\tau|^{(\varphi(2q)-2\varphi(q)} \leq \frac{\mathbb{E}A_{(\tau,1)}{}^{q}(t)A_{(\tau,1)}{}^{q}(s)}{c|\tau|^{2(q+\varphi(q))}} \leq |s-t-\tau|^{\varphi(2q)-2\varphi(q)}.$$

7.3 Proof of Theorem 2

Let us now consider the case $q > 1$, $q \notin \mathbb{N}$. Let us define $q = m + \varepsilon$ with $m = [q]$ and $0 < \varepsilon < 1$. One writes:

$$\mathbb{E}A_{(\tau,1)}{}^{q}(s)A_{(\tau,1)}{}^{q}(t) = \mathbb{E}A_{(\tau,1)}{}^{m-1}(s)A_{(\tau,1)}{}^{m-1}(t)A_{(\tau,1)}{}^{1+\varepsilon}(s)A_{(\tau,1)}{}^{1+\varepsilon}(t).$$

(if $m = 1$, only the term $\mathbb{E}A_{(\tau,1)}{}^{1+\varepsilon}(s)A_{(\tau,1)}{}^{1+\varepsilon}(t)$ appears, but the proof is the same).

Again, $A_{(\tau,1)}{}^{m-1}(s)A_{(\tau,1)}{}^{m-1}(t)$ can be written as a multiple integral. Moreover, a classical Hölder inequality yields:

$$A_{(\tau,1)}{}^{1+\varepsilon}(t) = \left(\int_t^{t+\tau} Q_r(x)dx\right)^{1+\varepsilon} \leq \tau^{\varepsilon} \int_t^{t+\tau} Q_r^{1+\varepsilon}(x)dx.$$

Hence, one gets

$$\mathbb{E}A_{(\tau,1)}{}^{q}(s)A_{(\tau,1)}{}^{q}(t) \leq \lim_{r\to 0} \tau^{2\varepsilon} \int_D E \prod_{i=1}^{m-1} Q_r(x_i)Q_r(y_i)Q_r(x_m)^{1+\varepsilon}Q_r(y_m)^{1+\varepsilon}d(x_1,\ldots,x_m)d(y_1,\ldots,y_m) \tag{28}$$

where $D = [t, t+\tau]^m \times [s, s+\tau]^m$. From Lemma 1, one can write

$$\mathbb{E}\prod_{i=1}^{m-1} Q_r(x_i)Q_r(y_i)Q_r(x_m)^{1+\varepsilon}Q_r(y_m)^{1+\varepsilon}$$

as the product of three terms. The term

$$J = e^{\sum_{j=1}^{q}\sum_{k=1}^{q}\beta(j+q,k)\rho_r(y_j,x_k)}$$

is bounded above by $|t-s+\tau|^{\varphi(2q)-2\varphi(q)}$ and the integral on D of the other terms is bounded by $\tau^{2\varphi(q)+2m}$. Finally, one gets

$$\mathbb{E}A_{(\tau,1)}{}^{q}(s)A_{(\tau,1)}{}^{q}(t) \leq \tau^{2\varepsilon}\tau^{2m+2\varphi(q)}|t-s+\tau|^{\varphi(2q)-2\varphi(q)} \leq \tau^{2q+2\varphi(2q)}|t-s+\tau|^{\varphi(2q)-2\varphi(q)}.$$

To obtain a lower bound, one writes

$$\mathbb{E}A_{(\tau,1)}{}^{q}(s)A_{(\tau,1)}{}^{q}(t) = \mathbb{E}A_{(\tau,1)}{}^{m}(s)A_{(\tau,1)}{}^{m}(t)A_{(\tau,1)}{}^{\varepsilon}(s)A_{(\tau,1)}{}^{\varepsilon}(t).$$

with

$$A_{(\tau,1)}{}^{\varepsilon}(t) = \left(\int_t^{t+\tau} Q_r(x)dx\right)^{\varepsilon} \geq \tau^{\varepsilon-1} \int_t^{t+\tau} Q_r^{\varepsilon}(x)dx.$$

With the same arguments than before, ones gets

$$\mathbb{E}A_{(\tau,1)}{}^{q}(s)A_{(\tau,1)}{}^{q}(t) \geq C(q)\tau^{2q+2\varphi(q)}|t-s+\tau|^{\varphi(2q)-2\varphi(q)}$$

where $C(q) > 0$ depends only on q.

7.4 Proof of Proposition 6

$$\begin{aligned}&\mathbb{E}\mathscr{B}^2_{(\tau,1)}(t)\mathbb{E}\mathscr{B}^2_{(\tau,1)}(s)\\&=\mathbb{E}[\mathbb{E}\left([B_H(u)-B_H(v)]^2[B_H(x)-B_H(y)]^2|u=A(s+\tau),v=A(s),x=A(t+\tau),y=A(t)\right)]\end{aligned}$$

Because $X=B_H(u)-B_H(v)$ and $Y=B_H(x)-B_H(y)$ are two gaussian vectors with finite variance, one can use the classical equality

$$\mathbb{E}X^2Y^2=\mathbb{E}X^2\mathbb{E}Y^2+2(\mathbb{E}XY)^2, \tag{29}$$

which leads by the Cauchy-Schwarz inequality to

$$\mathbb{E}X^2\mathbb{E}Y^2\le\mathbb{E}X^2Y^2\le 3\mathbb{E}X^2\mathbb{E}Y^2,$$

with $\mathbb{E}X^2=\mathbb{E}(B_H(u)-B_H(v))^2=\sigma^2|u-v|^{2H}$, $\mathbb{E}Y^2=\sigma^2|x-y|^{2H}$.

Therefore,

$$\sigma^4\mathbb{E}{A_{(\tau,1)}}^{2H}(t){A_{(\tau,1)}}^{2H}(s)\le\mathbb{E}\mathscr{B}^2_{(\tau,1)}(t)\mathbb{E}\mathscr{B}^2_{(\tau,1)}(s)\le 3\sigma^4\mathbb{E}{A_{(\tau,1)}}^{2H}(t){A_{(\tau,1)}}^{2H}(s)$$

which, combined to Theorem 2 with $q=2H$, gives the announced result.

7.5 Proof of Proposition 7

Because φ is chosen such that $q_c>4$, the quantity $I=\mathbb{E}{A_{(\tau,2)}}^2(t){A_{(\tau,2)}}^2(s)$ is finite. Developing ${A_{(\tau,2)}}^2(t){A_{(\tau,2)}}^2(s)=(A_{(\tau,1)}(t+\tau)-A_{(\tau,1)}(t))^2(A_{(\tau,1)}(s+\tau)-A(s))^2$ enables to rewrite the expectation as the sum of 9 terms. Each term can be written as the integral of a product of functions Q_r. More precisely, with $r>t-s+2\tau$,

$$I=4\lim_{r\to 0}\int_{u=t}^{t+\tau}\int_{v=u}^{t+\tau}\int_{x=s}^{s+\tau}\int_{y=x}^{s+\tau}J(u,v,x,y)dydxdvdu$$

with

$$\begin{aligned}J(u,v,x,y)=\;&F(u+\tau,v+\tau,x+\tau,y+\tau)-2F(u,v+\tau,x+\tau,y+\tau)\\&+F(u+\tau,v+\tau,x,y)-2F(u+\tau,v+\tau,x,y+\tau)\\&+4F(u,v+\tau,x,y+\tau)-2F(u,v+\tau,x,y)\\&+F(u,v,x+\tau,y+\tau)-2F(u,v,x,y+\tau)+F(u,v,x,y)\end{aligned}$$

where $F(a,b,c,d)=G(a,b)G(c,d)H(a,b,c,d)$ for $a\le b\le c\le d$ with

$$G(z_1,z_2)=G_r(z_1,z_2)=e^{\beta(2,1)\rho_r(z_2-z_1)}$$

and

$$\begin{aligned} H(a,b,c,d) &= e^{-\beta(3,2)\rho_r(c-b)} e^{\beta(3,1)\rho_r(c-a)} e^{\beta(4,2)\rho_r(d-b)} e^{\beta(1,4)\rho_r(d-a)} \\ &= (c-b)^{\beta(3,2)}(c-a)^{\beta(3,1)}(d-b)^{\beta(4,2)}(d-a)^{\beta(4,1)}. \end{aligned}$$

Hence, J can be written as

$$\begin{aligned} J = &-\Delta_x\Delta_v F(u,v,x,y+\tau) + \Delta_u\Delta_x F(u,v+\tau,x,y+\tau) \\ &+\Delta_v\Delta_y F(u,v,x,y) - \Delta_u\Delta_y F(u,v+\tau,x,y) \end{aligned}$$

where $\Delta_u F(u,v,x,y)$ (resp. Δ_v, Δ_x and Δ_y) denotes the forward difference $F(u+\tau,v,x,y) - F(u,v,x,y)$ (resp. $F(u,v+\tau,x,y) - F(u,v,x,y)$, etc).

Expressing J in terms of G and H, one finds

$$\begin{aligned} J = &\Delta_u G(u,v+\tau)\Delta_x G(x,y+\tau)H(u,v+\tau,x,y+\tau) \\ &+G(u,v+\tau)\Delta_x G(x,y+\tau)\Delta_u H(u,v+\tau,x,y+\tau) \\ &+\Delta_u G(u,v+\tau)G(x,y+\tau)\Delta_x H(u,v+\tau,x,y+\tau) \\ &+G(u,v+\tau)G(x,y+\tau)\Delta_u\Delta_x H(u,v+\tau,x,y+\tau) \\ &-\Delta_x G(x,y+\tau)\Delta_v G(u,v)H(u,v,x,y+\tau) - G(x,y+\tau)\Delta_v G(u,v)\Delta_x H(u,v,x,y+\tau) \\ &-\Delta_x G(x,y+\tau)G(u,v)\Delta_v H(u,v,x,y+\tau) - G(x,y+\tau)G(u,v)\Delta_x\Delta_v H(u,v,x,y+\tau) \\ &+\Delta_y G(x,y)\Delta_v G(u,v)H(u,v,x,y) + \Delta_y G(x,y)G(u,v)\Delta_v H(u,v,x,y) \\ &+G(x,y)\Delta_v G(u,v)\Delta_y H(u,v,x,y) + G(x,y)G(u,v)\Delta_v\Delta_y H(u,v,x,y) \\ &-\Delta_y G(x,y)\Delta_u G(u,v+\tau)H(u,v+\tau,x,y) - \Delta_y G(x,y)G(u,v+\tau)\Delta_u H(u,v+\tau,x,y) \\ &-G(x,y)\Delta_u G(u,v+\tau)\Delta_y H(u,v+\tau,x,y) - G(x,y)G(u,v+\tau)\Delta_u\Delta_v H(u,v+\tau,x,y). \end{aligned}$$

We first consider the terms where one or two finite differences are taken on H, for example, $G(x,y+\tau)\Delta_u G(u,v+\tau)\Delta_x H(u,v,x,y+\tau)$. One has, for any $0 < r < s-t-2\tau$,

$$|\Delta_x H(u,v,x,y+\tau)| \le \tau|t-s-2\tau|^{\varphi(4)-2\varphi(2)-1} + \tau R(\tau),$$

with $R(\tau) \to 0$ when $\tau \to 0$, $\Delta_v G(u,v) \le 0$ and $G(x,y+\tau) \ge 0$. Therefore,

$$\begin{aligned} A &= \int\int\int\int G(x,y+\tau)\Delta_u G(u,v+\tau)\Delta_x H(u,v,x,y+\tau)dydxdvdu \\ &\le \tau|t-s-2\tau|^{\varphi(4)-2\varphi(2)-1}\left(\int\int G(x,y+\tau))dxdy\right)\left(\int\int \Delta_v G(u,v)dudv\right). \end{aligned}$$

Besides, we have the following result whose proof is postponed to the end of this subsection.

Lemma 3. *There exists $c_1 > 0$ and $c_2 > 0$ such that*

$$\lim_{r\to 0}\int_{x=s}^{s+\tau}\int_{y=x}^{s+\tau} G(x,y+\tau)dydx \le \mathbb{E}A^2_{(2\tau,1)}(t) = c_1|\tau|^{\varphi(2)+2}$$

and

$$\lim_{r\to 0}\int_{u=t}^{t+\tau}\int_{v=u}^{t+\tau}\Delta_u G(u,v+\tau)dvdu = \mathbb{E}{A_{(\tau,2)}}^2(t)/4 = c_2|\tau|^{\varphi(2)+2}.$$

So,

$$\lim_{\tau\to 0}\frac{\int\int\int\int G(x,y+\tau)\Delta_v G(u,v)\Delta_v H(u,v,x,y+\tau)dudvdxdy}{|\tau|^{2(2+\varphi(2))}} = 0.$$

The other terms with finite difference on H can be dealt with in a similar way. Finally, since

$$\lim_{r\to 0}\Delta_u G(u,v+\tau) = -\lim_{r\to 0}\Delta_v G(u,v) = \lim_{r\to 0}\Delta_x G(x,y+\tau) = -\lim_{r\to 0}\Delta_x G(x,y) = 1/4\mathbb{E}{A_{(\tau,2)}}^2(t),$$

and since

$$|s-t+2\tau|^{\varphi(4)-2\varphi(2)} \le H(u,v+\tau,x,y+\tau) \le |s-t-2\tau|^{\varphi(4)-2\varphi(2)},$$

$$|s-t+2\tau|^{\varphi(4)-2\varphi(2)} \le H(u,v+\tau,x,y) \le |s-t-2\tau|^{\varphi(4)-2\varphi(2)},$$

we obtain the announced result.

Let us now give the proof of Lemma 3. The first inequality is trivial since we have

$$\begin{aligned}\mathbb{E}A^2_{(2\tau,1)}(t) &= \lim_{r\to 0}\int_t^{t+2\tau}\int_t^{t+2\tau}\mathbb{E}Q_r(x)Q_r(y)dxdy\\ &\ge \int_t^{t+\tau}\int_t^{t+\tau}\mathbb{E}Q_r(x)Q_r(y+\tau)dxdy\\ &= \int_t^{t+\tau}\int_t^{t+\tau}G(x,y+\tau)dxdy.\end{aligned}$$

For the second point, let us remark first, that

$$\begin{aligned}\mathbb{E}{A_{(\tau,2)}}^2(t) &= (A_{(\tau,1)}(t+\tau)-A_{(\tau,1)}(t))^2\\ &= A_{(\tau,1)}(t+\tau)^2 - 2A_{(\tau,1)}(t)A_{(\tau,1)}(t+\tau) + {A_{(\tau,1)}}^2(t)\\ &= \lim_{r\to 0}\int_t^{t+\tau}\int_t^{t+\tau}(\mathbb{E}Q_r(x+\tau)Q_r(y+\tau) - 2\mathbb{E}Q_r(x)Q_r(y+\tau)\\ &\quad +\mathbb{E}Q_r(x)Q_r(x))dxdy).\end{aligned}$$

But, $\mathbb{E}Q_r(x+\tau)Q_r(y+\tau) = \mathbb{E}Q_r(x)Q_r(y)$ and

$$\begin{aligned}\mathbb{E}{A_{(\tau,2)}}^2(s) &= 2\lim_{r\to 0}\int_t^{t+\tau}\int_t^{t+\tau}(\mathbb{E}Q_r(x+\tau)Q_r(y+\tau) - \mathbb{E}Q_r(x)Q_r(y+\tau))dxdy\\ &= 4\lim_{r\to 0}\int_{x=s}^{t+\tau}\int_{y=x}^{t+\tau}\Delta_x G(x,y+\tau)dxdy.\end{aligned}$$

We still have to show that $\lim_{r\to 0}\int\int\Delta_u G(u,v+\tau)dudv = c\tau^{\varphi(2)+2}$. First, note that

$$\lim_{r\to 0}\int_{u=t}^{t+\tau}\int_{v=u}^{t+\tau}\Delta_u G(u,v+\tau)dudv = \lim_{r\to 0}\int_{u=t}^{t+\tau}\int_{v=u}^{t+\tau}(G(u+\tau,v+\tau)-G(u,v+\tau))dudv.$$

But,

$$\lim_{r\to 0}\int_{u=t}^{t+\tau}\int_{v=u}^{t+\tau}G(u+\tau,v+\tau)dudv = 1/2\lim_{r\to 0}\int_{u=t}^{t+\tau}\int_{v=t}^{t+\tau}\mathbb{E}Q_r(u)Q_r(v)dudv = 1/2\mathbb{E}{A_{(\tau,1)}}^2(t) = c|\tau|^{\varphi(2)+2}$$

for some $c > 0$ and

$$\lim_{r\to 0}\int_{u=t}^{t+\tau}\int_{v=u}^{t+\tau}G(u,v+\tau) = \lim_{r\to 0}\int_{u=t}^{t+\tau}\int_{v=u}^{t+\tau}|v+\tau-u|^{\varphi(2)}dvdu = \widetilde{c}|\tau|^{\varphi(2)+2}.$$

for some $\widetilde{c} > 0$. Thus,

$$\lim_{r\to 0}\int_{u=t}^{t+\tau}\int_{v=u}^{t+\tau}\Delta_u G(u,v+\tau)dvdu = C|\tau|^{\varphi(2)+2}$$

for a $C \in \mathbb{R}$. Since $\lim_{r\to 0}\int\int \Delta_u G(u,v)dudv = \mathbb{E}{A_{(\tau,2)}}^2(s)/4 > 0$, it comes that $C > 0$ and Lemma 3 is proven.

Remark 2: The crucial point in the above proof consists of the fact that when $q \geq 2$, the covariance function can be split into a number of terms, expressed with auto-terms G and cross terms H (whose that involve the dependence in $|t-s|$). Increments - of any order P - for some of these terms apply only to the auto-terms G and hence do not produce in reduction the in the rate of decrease in $|t-s|$ (only related to the cross terms H). A contrario, for $q = 1$, $G \equiv 1$ and the increments are taken on H, leading to Formula (20). This qualitative argument indicates that P does not play any role in the control of higher order correlation functions and is founding for the formulation of Conjectures 1 and 2 in Section 5.

Remark 3: Also, it is worth mentioning that in this derivation of the proof of Proposition 6, one obtains a number of higher order terms in the expansion in $|\tau|$, whose impact is significant, so that the leading power law term may be difficult to observe practically (cf. Discussion in Section 5.2.3).

7.6 *Proof of Proposition 8*

$$\begin{aligned}
&\mathbb{E}\mathscr{B}^2_{(\tau,2)}(t)\mathbb{E}\mathscr{B}^2_{(\tau,2)}(s)\\
&= \mathbb{E}\left(\mathbb{E}[B_H(w)-2B_H(v)+B_H(u))]^2[B_H(z)-2B_H(y)+B_H(x)]^2|\right.\\
&\quad w=A(s+2\tau), v=A(s+\tau), u=A(s), z=A(t+2\tau), y=A(t+\tau), x=A(t))
\end{aligned}$$

where $X = B_H(w)-2B_H(v)+B_H(u)$ and $Y = B_H(z)-2B_H(y)+B_H(x)$ are two gaussian vectors with finite variance. Equality (29) and Cauchy-Schwarz inequality yield

$$\mathbb{E}X^2\mathbb{E}Y^2 \le \mathbb{E}X^2Y^2 \le 3\mathbb{E}X^2\mathbb{E}Y^2$$

with

$$\mathbb{E}X^2 = \mathbb{E}(B_H(w) - 2B_H(v) + B_H(u))^2 = \sigma^2(2|w-v|^{2H} - |w-u|^{2H} + 2|v-u|^{2H}).$$

In addition, since A is a non decreasing function,

$$2|A(t+2\tau)-A(t+\tau)|^{2H} - |A(t+2\tau)-A(t)|^{2H} + 2|A(t+\tau)-A(t)|^{2H} \le 2A_{(\tau,1)}{}^{2H}(t+\tau) + 2A_{(\tau,1)}{}^{2H}(t) \le 2A^{2H}_{(2\tau,1)}(t)$$

and there exists $C_H > 0$ such that

$$2|A(t+2\tau)-A(t+\tau)|^{2H} + |A(t+2\tau)-A(t)|^{2H} + 2|A(t+\tau)-A(t)|^{2H} \ge C_H(A_{(\tau,1)}(t+\tau) + A_{(\tau,1)}(t)) \ge C_H A_{(\tau,1)}(t).$$

Finally, it comes

$$C_H\sigma^4\mathbb{E}A_{(\tau,1)}{}^{2H}(t)A_{(\tau,1)}{}^{2H}(s) \le \mathbb{E}\mathscr{B}^2_{(\tau,2)}(t)\mathbb{E}\mathscr{B}^2_{(\tau,2)}(s) \le 4\sigma^4\mathbb{E}A^{2H}_{(2\tau,1)}(t)A^{2H}_{(2\tau,1)}(s),$$

which, combined with Theorem 2, leads to the result.

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