On a Bayesian framework for the multifractal analysis of multivariate data

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University of Pisa, 27 Feb. 2017

Multifractal spectrum

- Local regularity of $X(t)$ at t_0

  Hölder exponent

  $h(t_0) = \sup_\alpha \{\alpha : \ |X(t) - X(t_0)| < C|t - t_0|^\alpha\}$ $\qquad 0 < \alpha$

  $h(t_0) \to 1 \Rightarrow$ smooth, very regular,
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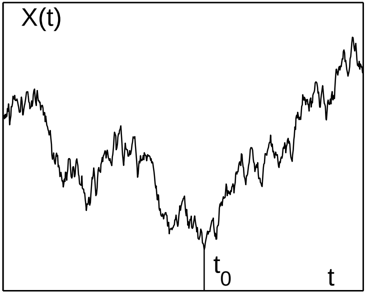

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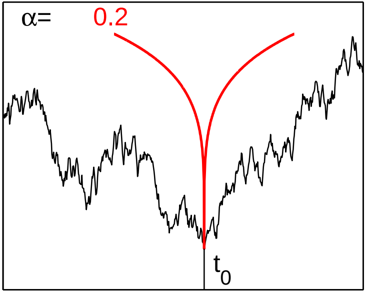

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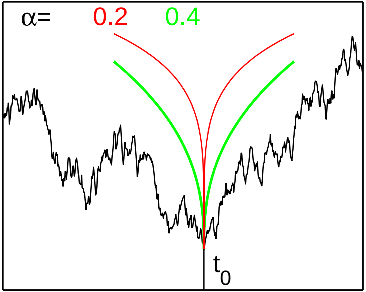

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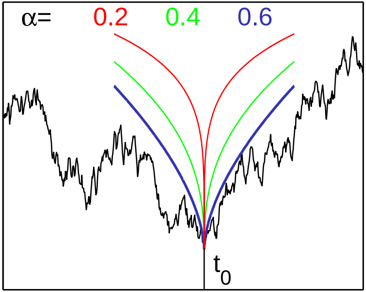

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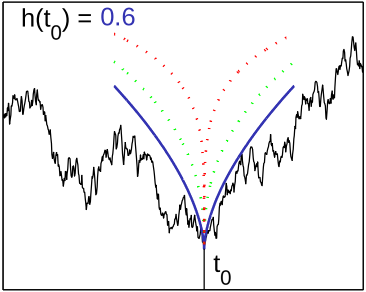

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  - Set of points that share same regularity $\{t_i|h(t_i) = h\}$
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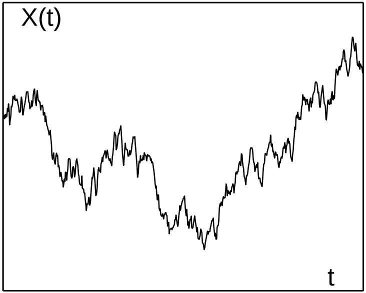

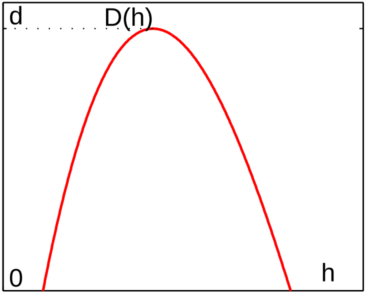

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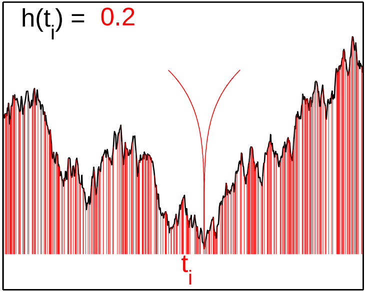

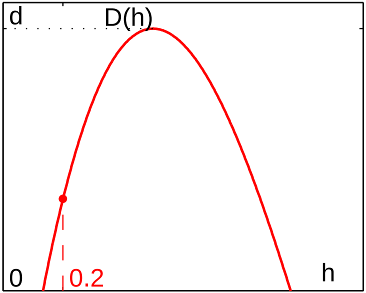

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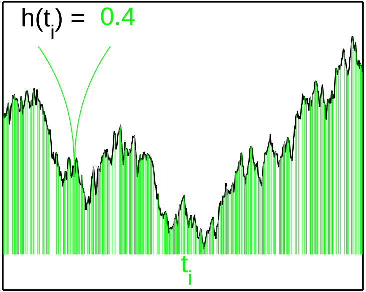

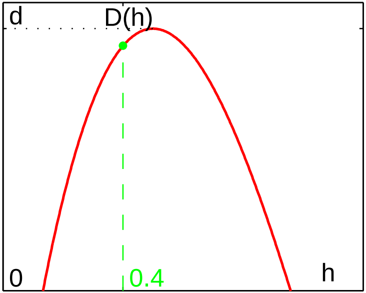

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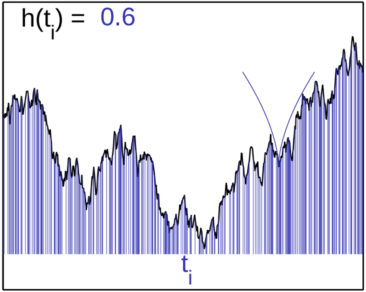

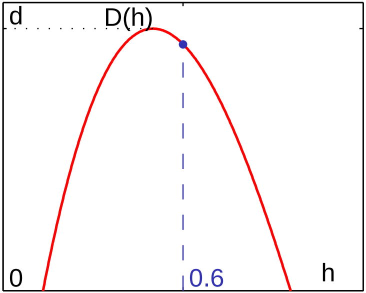

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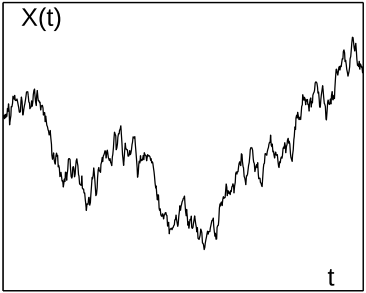
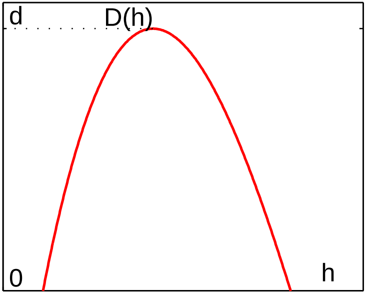
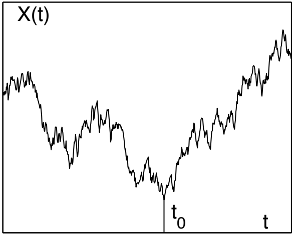


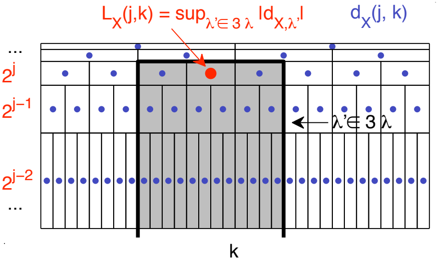


Multifractal formalism

- $D(h)$ in practice $\rightarrow$ multifractal formalism [Parisi85]
- Multiresolution quantities: wavelet leaders $\{\ell(j\cdot,\cdot)\}$ [Jaffard04]

$$\ell(j,k) = \sup_{\lambda' \subset 3\lambda_{j,k}} |d(\lambda')|, \qquad d(j,k): \text{ DWT coefficient}$$





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  - $\sim$ fluctuations of regularity
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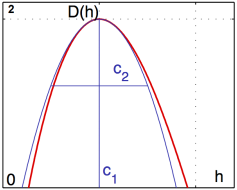

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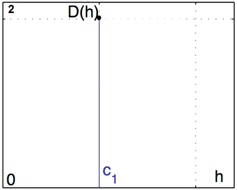

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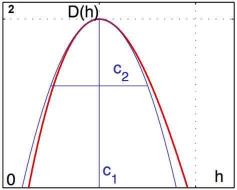
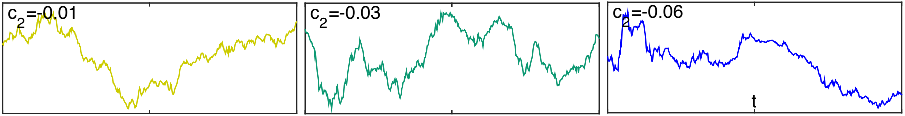
  - multifractal multiplicative cascades $\rightarrow$ $c_2 < 0$



Estimation of the multifractality parameter

- Estimation of c_2 is challenging
  - linear regression-based estimation [Castaing93]
    - X *poor estimation performance* $\longrightarrow$ *need (very) long time series*
  - existing alternatives unsatisfactory (fully parametric models, ...)

Synthetic multiplicative cascades with different c_2



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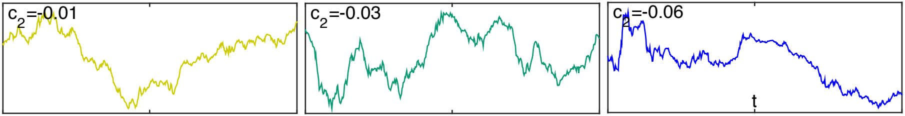

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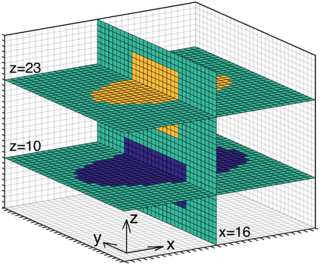
2. Bayesian estimation for c_2 for multivariate data [IWSSIP16,EUSIPCO16,ICIP16]
   - regularization using Markov field joint prior
   - $\rightarrow$ *further reduced variance*
   - $\rightarrow$ *computational cost* $\sim$ *linear regression*



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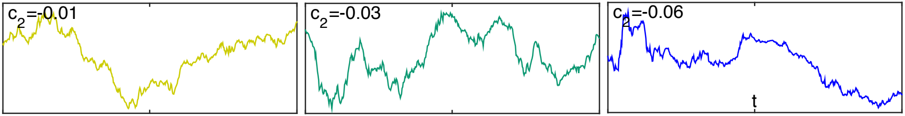
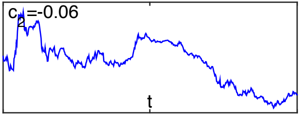


Bayesian model for single time series



Marginal distribution of log-leaders

- Marginal distribution of log-leaders well approximated by Gaussian [ICASSP13,TIP15]

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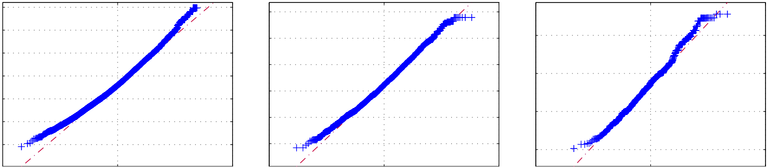

canonical Mandelbrot cascades with log-Normal multipliers

[Yaglom66,Mandelbrot74]

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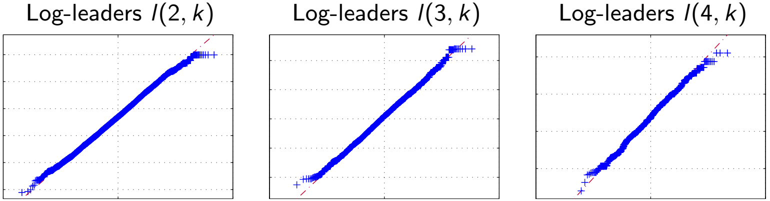

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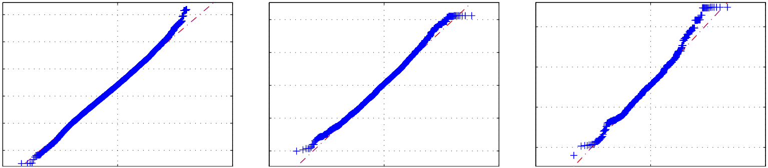

compound Poisson cascades with log-Normal multipliers

[Barral02,Chainais07]

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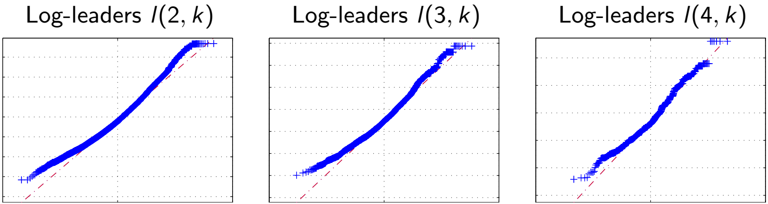

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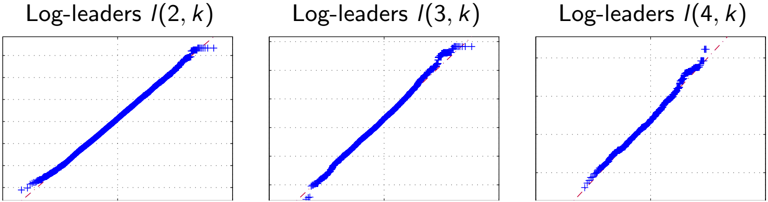

multifractal random walk (MRW)

[Bacry01,Robert10]

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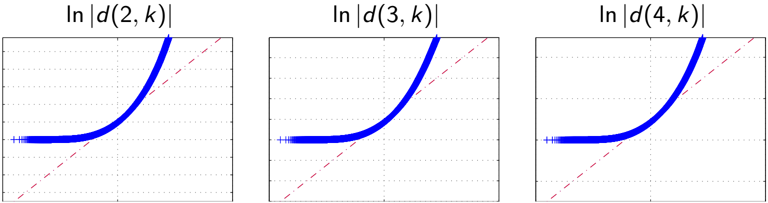
  model NOT valid for (log-)wavelet coefficients $\ln|d(j,k)|$



multifractal random walk (MRW)

[Bacry01,Robert10]

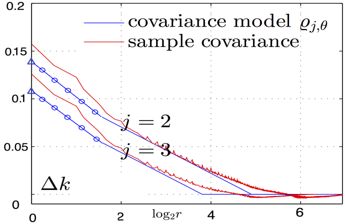
Gaussian random field parametric model

- Mean

$$\boxed{\mathbb{E}[l(j,k)] = c_1^0 + j c_1 \ln 2}$$ (discarded below)

- Variance-covariance $\rightarrow$ piecewise logarithmic model $\varrho_{j,\theta}(\Delta k)$
  - parameters $\theta = [\theta_1, \theta_2]^T = [c_2, c_2^0]^T$

    $\longrightarrow$ generic property for multifractal multiplicative cascades



$$\mathrm{Cov}[l(j,k), l(j,k+\Delta k)] \approx$$

$$\varrho_{j,\theta}(\Delta k) = \begin{cases} c_2^0 + c_2 j \ln 2 & \Delta k = 0 \\ \varrho_j^{(0)}(|\Delta k|;\theta) & 0 \le |\Delta k| \le 3 \\ \varrho_j^{(1)}(|\Delta k|;\theta) = \max\left(0, C_{IS} + c_2 \ln 2^j |\Delta k|\right) & 3 \le |\Delta k| \end{cases}$$

From a standard likelihood...

- Likelihood w.r.t. θ (sample) mean removed
  - log-leaders at scale j, $\boldsymbol{l}_j = (l(j,1), l(j,2), \dots)$

$$p(\boldsymbol{l}_j|\theta) \propto (\det \Sigma_{j,\theta})^{-\frac{1}{2}} e^{-\frac{1}{2}\boldsymbol{l}_j^T \Sigma_{j,\theta}^{-1} \boldsymbol{l}_j}$$

    - $\Sigma_{j,\theta}$ covariance matrix induced by parametric model $\varrho_{j,\theta}(\Delta k)$
  - collection of log-leaders $j = j_1, \dots, j_2$, $\boldsymbol{l} = [\boldsymbol{l}_{j_1}^T, ..., \boldsymbol{l}_{j_2}]^T$

$\rightarrow$ interscale independence assumption

$$p(\boldsymbol{l}|\theta) \propto \prod_{j=j_1}^{j_2} (\det \Sigma_{j,\theta})^{-\frac{1}{2}} e^{-\frac{1}{2}\boldsymbol{l}_j^T \Sigma_{j,\theta}^{-1} \boldsymbol{l}_j}$$

X inversion of $\Sigma_{j,\theta}$ prohibitive $\rightarrow$ Whittle approximation
X constraints on θ ($\Sigma_{j,\theta}$ p.d.) $\rightarrow$ reparametrization
X conjugacy of priors for θ $\rightarrow$ data augmentation

From a standard likelihood...

- Likelihood w.r.t. θ (sample) mean removed
  - log-leaders at scale j, $\boldsymbol{l}_j = (l(j,1), l(j,2), \dots)$

$$p(\boldsymbol{l}_j|\theta) \propto (\det \Sigma_{j,\theta})^{-\frac{1}{2}} e^{-\frac{1}{2}\boldsymbol{l}_j^T \Sigma_{j,\theta}^{-1} \boldsymbol{l}_j}$$

  - $\Sigma_{j,\theta}$ covariance matrix induced by parametric model $\varrho_{j,\theta}(\Delta k)$
  - collection of log-leaders $j = j_1, \dots, j_2$, $\boldsymbol{l} = [\boldsymbol{l}_{j_1}^T, ..., \boldsymbol{l}_{j_2}]^T$

$\rightarrow$ interscale independence assumption

$$p(\boldsymbol{l}|\theta) \propto \prod_{j=j_1}^{j_2} (\det \Sigma_{j,\theta})^{-\frac{1}{2}} e^{-\frac{1}{2}\boldsymbol{l}_j^T \Sigma_{j,\theta}^{-1} \boldsymbol{l}_j}$$

X inversion of $\Sigma_{j,\theta}$ prohibitive	$\rightarrow$ Whittle approximation
X constraints on θ ($\Sigma_{j,\theta}$ p.d.)	$\rightarrow$ reparametrization
X conjugacy of priors for θ	$\rightarrow$ data augmentation

... to a Data Augmented Likelihood

[TIP15,ICASSP16]

- Whittle approximation $\Longrightarrow$ Fourier transform (DFT) of centered log-leaders $\boldsymbol{l}_j$

$$\boldsymbol{y}_j = DFT(\boldsymbol{l}_j)$$

- Reparametrization $\Longrightarrow$ independent positivity constraints on parameters

$$\boldsymbol{v} = \psi((c_2, c_2^0)) \in \mathbb{R}_\star^{+2}$$

- Data augmentation $\Longrightarrow$ hidden mean $\boldsymbol{\mu}_j$ for $\boldsymbol{y}_j$

$\Longrightarrow$ complex Gaussian model for $\boldsymbol{y} = [\boldsymbol{y}_{j_1}^T, ..., \boldsymbol{y}_{j_2}^T]^T$

$$\begin{cases} \boldsymbol{y}|\boldsymbol{\mu}, v_2 & \sim \mathcal{CN}(\boldsymbol{\mu}, v_2 \boldsymbol{F}_2) \quad \text{observed data} \\ \boldsymbol{\mu}|v_1 & \sim \mathcal{CN}(\boldsymbol{0}, v_1 \boldsymbol{F}_1) \quad \text{hidden mean} \end{cases}$$

$\mathbf{F}_1$, $\mathbf{F}_2$ diagonal, positive definite, known and fixed

$$p(\boldsymbol{y}, \boldsymbol{\mu}|\boldsymbol{v}) \propto p(\boldsymbol{y}|\boldsymbol{\mu}, v_2)\, p(\boldsymbol{\mu}|v_1)$$

$$\propto {v_2}^{-N_Y} \exp\left(-\frac{1}{v_2}(\boldsymbol{y}-\boldsymbol{\mu})^H \boldsymbol{F}_2^{-1}(\boldsymbol{y}-\boldsymbol{\mu})\right) \times {v_1}^{-N_Y} \exp\left(-\frac{1}{v_1}\boldsymbol{\mu}^H \boldsymbol{F}_1^{-1}\boldsymbol{\mu}\right)$$

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Augmented likelihood based Bayesian model [ICASSP16]

- Augmented likelihood

$$p(\boldsymbol{y}, \boldsymbol{\mu}|\boldsymbol{v}) \propto {v_2}^{-N_Y} \exp\left(-\frac{1}{v_2}(\boldsymbol{y}-\boldsymbol{\mu})^H \boldsymbol{F}_2^{-1}(\boldsymbol{y}-\boldsymbol{\mu})\right) \times {v_1}^{-N_Y} \exp\left(-\frac{1}{v_1}\boldsymbol{\mu}^H \boldsymbol{F}_1^{-1}\boldsymbol{\mu}\right)$$

- Prior
  v_i as variance of Gaussian $\rightarrow$ conjugate inverse-gamma prior $\mathcal{IG}(\alpha_i, \beta_i)$
- Posterior

$$p(\boldsymbol{v}, \boldsymbol{\mu}|\boldsymbol{y}) \propto p(\boldsymbol{y}, \boldsymbol{\mu}|\boldsymbol{v}) p(v_1) p(v_2)$$

- Bayesian estimators via MCMC algorithm
  $\rightarrow$ marginal posterior mean estimator (MMSE) $\boldsymbol{v}^{\text{MMSE}} = \mathbb{E}[\boldsymbol{v}|\boldsymbol{y}]$

Markov Chain Monte Carlo Algorithm

- Sampling of $\boldsymbol{\mu}$ and parameters $\boldsymbol{v}$

  $p(\boldsymbol{\mu}|\boldsymbol{v}, \boldsymbol{y})$ closed-form Gaussian distribution

  $p(\boldsymbol{v}_i|\boldsymbol{v}_{j\neq i}, \boldsymbol{\mu}, \boldsymbol{y})$ closed-form inverse-gamma distributions

  all standard distributions $\rightarrow$ no Metropolis-Hasting moves

- Performance for synthetic data *(further details later)*
  - $N = 512$, $c_2 = -0.01, \ldots, -0.08$
  - estimation performance improved by factor up to ~ 4
  - about 5 to 2 times slower than linear regression

	LF	IG
|b|	0.0158	0.0051
std	0.0800	0.0255
rmse	0.0819	0.0262

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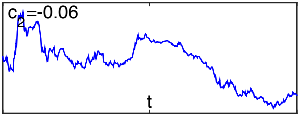
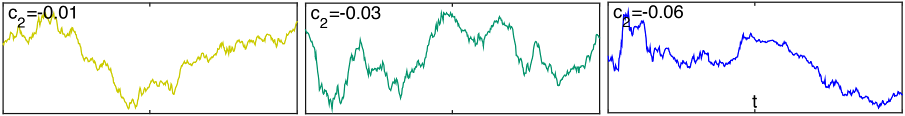
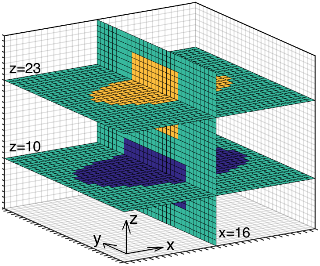
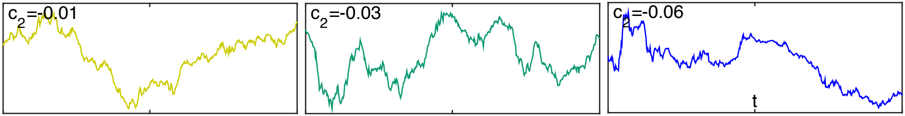



c2=-0.01
c2=-0.03
c2=-0.06
t

Bayesian model for multivariate time series

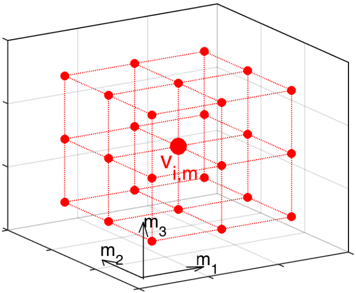


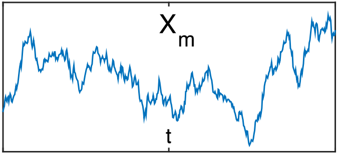


Strategy: Hierarchical Bayesian models

for volumetric time series (voxels), $X_{\boldsymbol{m}}$, $\boldsymbol{m} \triangleq (m_1, m_2, m_3)$, of length N
(other data structures possible)

1. Statistical model $p(\boldsymbol{y}_{\boldsymbol{m}}, \boldsymbol{\mu}_{\boldsymbol{m}} | \boldsymbol{v}_{\boldsymbol{m}})$
   - $\boldsymbol{y}_{\boldsymbol{m}}$: Fourier coeff's of log-leaders of $X_{\boldsymbol{m}}$
   - $\boldsymbol{\mu}_{\boldsymbol{m}}$: latent variables
   - $\boldsymbol{v}_{\boldsymbol{m}}$: parameter vector





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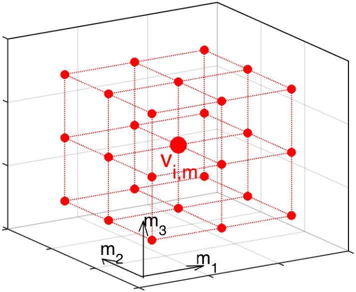
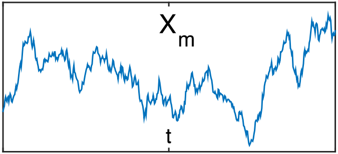


2. Prior independence between voxels

$$p(\boldsymbol{Y}, \boldsymbol{M} | \boldsymbol{V}) \propto \prod_{\boldsymbol{m}} p(\boldsymbol{y_m}, \boldsymbol{\mu_m} | \boldsymbol{v_m})$$

   - $\boldsymbol{Y} \triangleq \{\boldsymbol{y_m}\}$
   - $\boldsymbol{M} \triangleq \{\boldsymbol{\mu_m}\}$
   - $\boldsymbol{V} \triangleq \{\boldsymbol{V}_1, \boldsymbol{V}_2\}$ ($\boldsymbol{V}_i \triangleq \{\theta_{i,\boldsymbol{m}}\}$, $i = 1, 2$)



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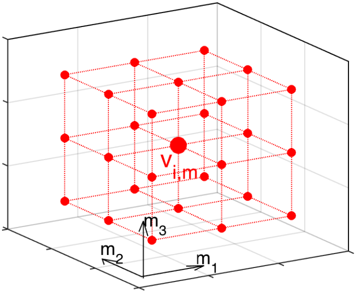

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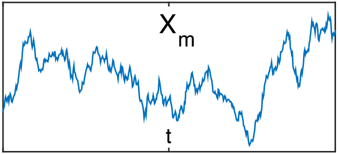

3. Design of regularizing priors on $\boldsymbol{V}$

Prior: joint Gamma Markov random field (GaMRF)

→ smooth evolution of multifractal parameters **v** (i.e., variances of Gaussians)

- Positive auxiliary variables $\boldsymbol{Z} = \{\boldsymbol{Z}_1, \boldsymbol{Z}_2\}$, $\boldsymbol{Z}_i = \{z_{i,\boldsymbol{m}}\}$
  ⟶ **induce dependence** between neighbouring elements of $\boldsymbol{V}_i$

- $v_{i,\boldsymbol{m}}$: connected to 8 variables $z_{i,\boldsymbol{m}'} \in \mathcal{V}_v(\boldsymbol{m})$
  $\mathcal{V}_v(\boldsymbol{m}) \triangleq \{\boldsymbol{m} + (i_1, i_2, i_3)\}_{i_1,i_2,i_3=0,1}$

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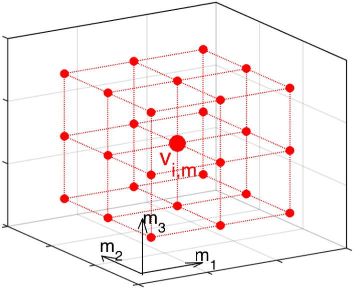

- Associated density [Dikmen10]

$$p(\boldsymbol{V}_i, \boldsymbol{Z}_i | \rho_i) \propto \prod_{\boldsymbol{m},n} \mathrm{e}^{(8\rho_i - 1)\log z_{i,\boldsymbol{m}}} \, \mathrm{e}^{-(8\rho_i+1)\log v_{i,\boldsymbol{m}}} \times \mathrm{e}^{-\frac{\rho_i}{v_{i,\boldsymbol{m}}} \sum_{\boldsymbol{m}' \in \mathcal{V}_v \boldsymbol{m}} z_{i,\boldsymbol{m}'}}$$

$$z_{i,\boldsymbol{m}} | \boldsymbol{V}_i \sim \mathcal{G}\big(8\rho_i, \big(\rho_i \sum_{\boldsymbol{m}' \in \mathcal{V}_z(\boldsymbol{m})} v_{i,\boldsymbol{k}'}^{-1}\big)^{-1}\big) \quad \rightarrow \text{gamma conditionals}$$

$$\boldsymbol{v}_{i,\boldsymbol{m}} | \boldsymbol{Z}_i \sim \mathcal{IG}\big(8\rho_i, \rho_i \sum_{\boldsymbol{m}' \in \mathcal{V}_v(\boldsymbol{m})} z_{i,\boldsymbol{m}'}\big) \quad \rightarrow \text{inverse-gamma cond.}$$

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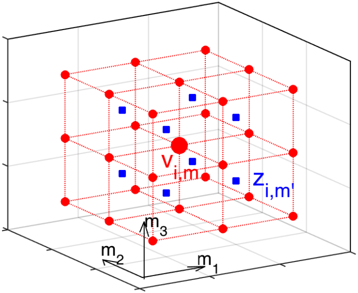

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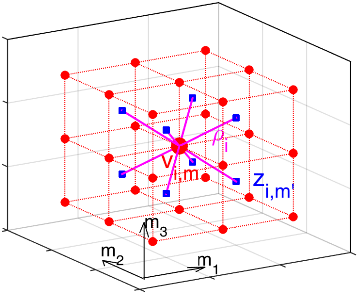

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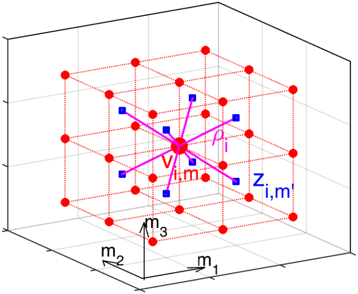

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Bayesian model

- Posterior distribution

$$p(\boldsymbol{V}, \boldsymbol{Z}, \boldsymbol{M} | \boldsymbol{Y}, \rho_1, \rho_2) \propto \underbrace{p(\boldsymbol{Y} | \boldsymbol{V}_2, \boldsymbol{M})\ p(\boldsymbol{M} | \boldsymbol{V}_1)}_{\textit{augmented likelihood}} \times \underbrace{p(\boldsymbol{V}_1, \boldsymbol{Z}_1 | \rho_1)\ p(\boldsymbol{V}_2, \boldsymbol{Z}_2 | \rho_2)}_{\textit{independent GaMRF priors}}$$

- Bayesian estimator → marginal posterior mean

$$\boldsymbol{V}_i^{\mathrm{MMSE}} = \mathbb{E}[\boldsymbol{V}_i | \boldsymbol{Y}, \rho_i] \approx (N_{mc} - N_{bi})^{-1} \sum_{q=N_{bi}}^{N_{mc}} \boldsymbol{V}_i^{(q)}$$

with $\{\boldsymbol{V}^{(q)}, \boldsymbol{Z}^{(q)}, \boldsymbol{M}^{(q)}\}_{q=0}^{N_{mc}}$ generated via MCMC algorithm [Robert05]

- Hyperparameters ρ_i not estimated here, fixed manually

Gibbs sampler: independent $\mathcal{IG}$ priors (univariate)

- Sampling of $\boldsymbol{M}$ and parameters $\boldsymbol{V}$

  $p(\boldsymbol{\mu_m} | \boldsymbol{V}, \boldsymbol{Y})$ closed-form Gaussian distribution

  $p(\boldsymbol{v}_{i,\boldsymbol{m}} | \boldsymbol{V}_{j \neq i}, \boldsymbol{M}, \boldsymbol{Y})$ closed-form inverse-gamma distributions

all standard distributions → no Metropolis-Hasting moves

→ efficient sampling scheme, tailored for large datasets

Gibbs sampler: joint GaMRF prior

- Sampling of $\boldsymbol{M}$ and parameters $\boldsymbol{V}$
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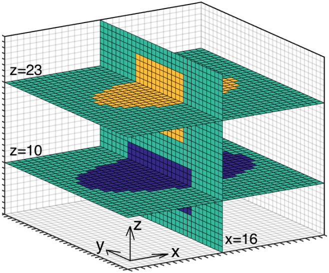
- Sampling of auxiliary variables $\boldsymbol{Z}$
  $p(z_{i,\boldsymbol{m}} | \boldsymbol{V}, \boldsymbol{M}, \boldsymbol{Y}, \boldsymbol{\rho})$ closed-form gamma distributions

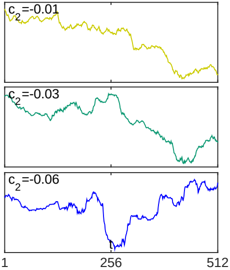
all standard distributions $\rightarrow$ no Metropolis-Hasting moves

$\rightarrow$ efficient sampling scheme, tailored for large datasets

Numerical simulations

- Synthetic multifractal time series: Multifractal Random Walk
  $\sim$ Mandelbrot's celebrated multiplicative cascades
- cube of 32^3 voxels of length $N = 512$
  - 3 zones with constant $c_2 \in \{-0.01, -0.03, -0.06\}$

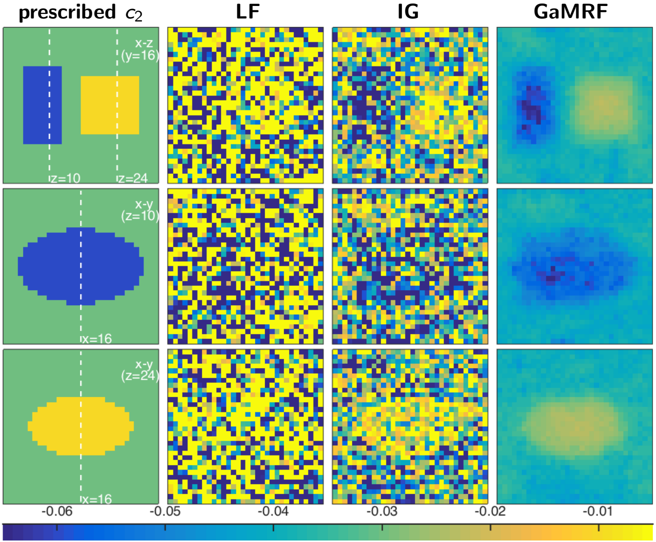




- Comparison of estimators for c_2 $(N_\psi = 2,\ j \in [2, 4])$
  - LF – univariate linear regression based estimation
  - IG – univariate Bayesian estimation
  - GaMRF – joint Bayesian estimator

Illustration for single realization: estimates

estimates for c_2



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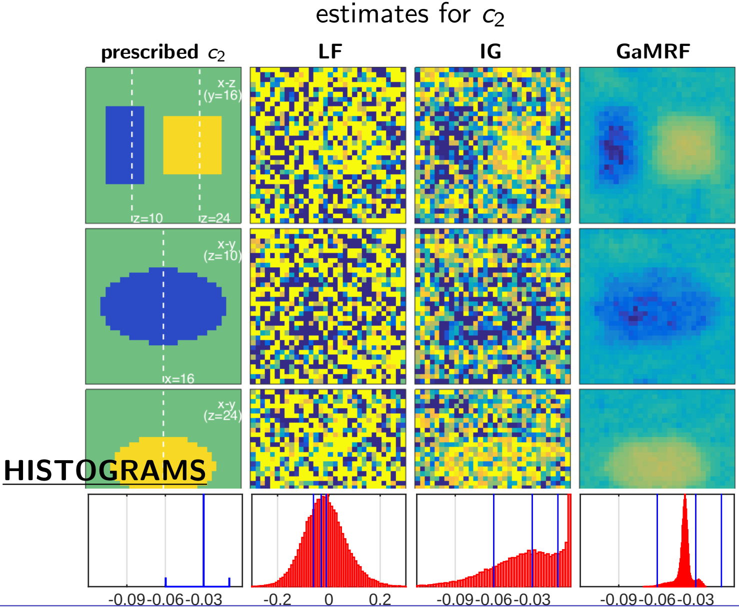
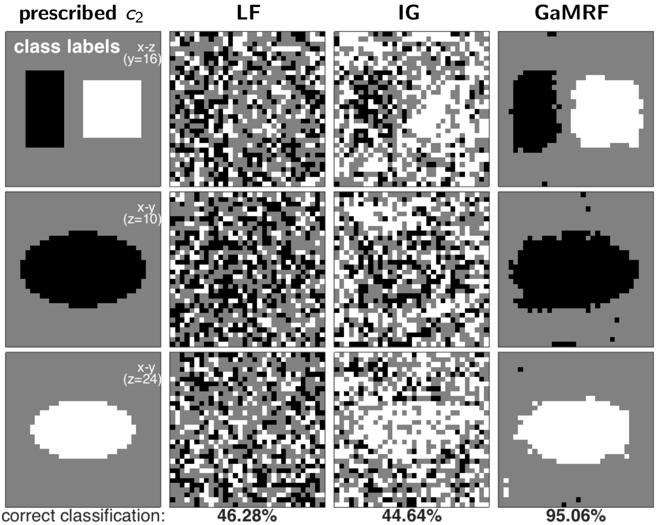


Illustration for single realization: histogram thresholding

k-means classification



Estimation performance

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std	0.0800	0.0255	0.0020
rmse	**0.0819**	**0.0262**	**0.0094**

$$\mathrm{b} = \widehat{\mathbb{E}}[\hat{c}_2] - c_2, \quad \mathrm{std} = \sqrt{\widehat{\mathsf{Var}}[\hat{c}_2]}, \quad \mathrm{rmse} = \sqrt{\mathrm{b}^2 + \mathrm{std}^2}$$

(100 independent realizations)

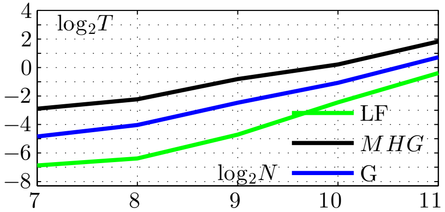
Estimation performance

	LF	IG	GaMRF
|b|	0.0158	0.0051	0.0092
std	0.0800	0.0255	0.0020
rmse	**0.0819**	**0.0262**	**0.0094**

$$\mathrm{b} = \widehat{\mathbb{E}}[\hat{c}_2] - c_2, \quad \mathrm{std} = \sqrt{\widehat{\mathrm{Var}}[\hat{c}_2]}, \quad \mathrm{rmse} = \sqrt{\mathrm{b}^2 + \mathrm{std}^2}$$

(100 independent realizations)

Computation time:



fMRI data experiment (with P. Ciuciu, CEA NeuroSpin, Paris)

Experimental design: verbal n-back working memory task ($n = 3$).

– serially presented upper-case letters (displayed 1s, separation 2s)
  → determine whether letter is same as that presented 3 stimuli before
– each run: alternating sequence of 8 blocks

Data acquisition.

– fMRI data acquisition at 3 Tesla (Siemens Trio, Germany).
– multi-band GE-EPI (TE=30ms, TR=1s, FA=61, b=2) sequence (CMRR, USA), 3-mm isotropic resolution, FOV of $192 \times 192 \times 144\text{mm}^3$
– resting-state fMRI images: participant at rest, with eyes closed
– 543 scans (9min10s) / 512 scans (8min39s) for resting state / task

Analysis setting.

– $N_\psi = 2$
– $j = [2, 5]$
– $N_{mc} = 16000$

– **regularization parameter**: $\rho = 1$ (preliminary analysis)

– shown here: single subject (arbitrarily chosen from 40 participants)

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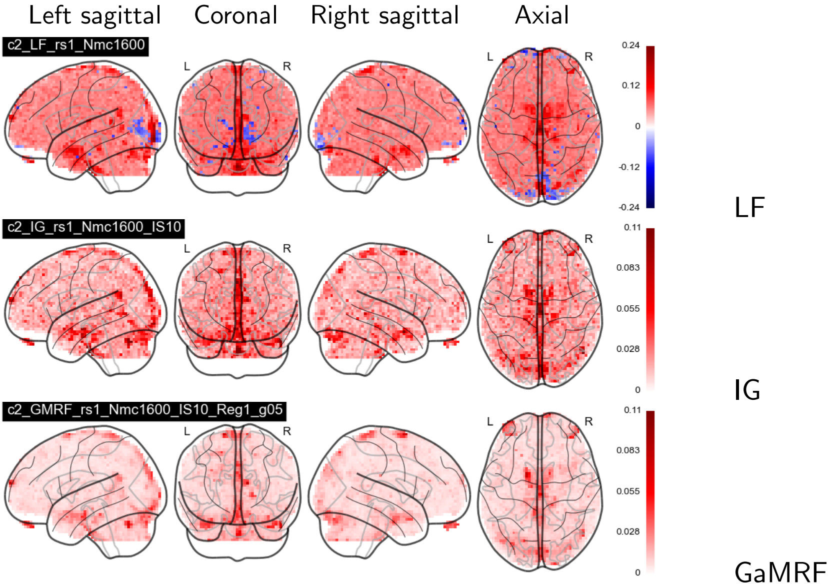
Numerical illustration for fMRI data

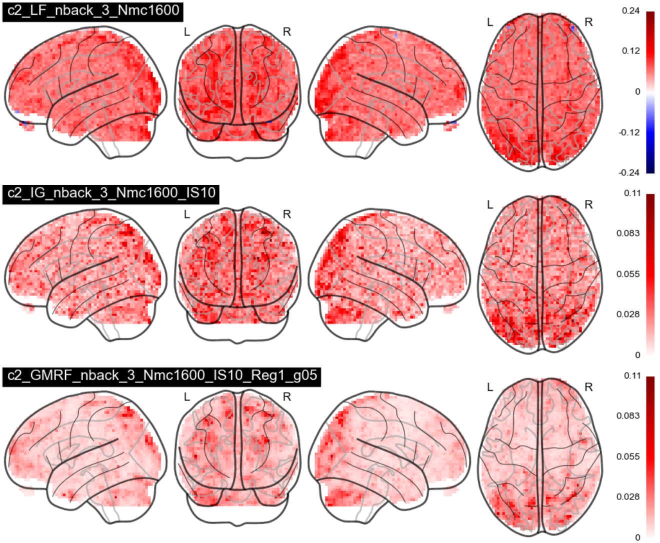
Resting state $(-c_2)$-maps



$\longrightarrow$ significant multifractality in default mode network.

Task (3-back run) $(-c_2)$-maps

Left sagittal Coronal Right sagittal Axial



LF

IG

GaMRF

$\longrightarrow$ increased multifractality. working memory network. occipital cortex.

Conclusions

- Bayesian estimation for c_2 of multivariate time series
  - hierarchical Bayesian model with smoothing priors:

$$\begin{cases} \text{data augmented Fourier domain likelihood} & (\sim \mathcal{CN}) \\ \text{GaMRF joint prior for } c_2 \text{ of different data components} \end{cases}$$

  - efficient inference via a Gibbs sampler

  → significantly improved estimation performance (gain: factor ~ 10)

- Alternative regularization for c_2 (not shown here)
  - simultaneous autoregression (SAR) smoothing prior
  - enables sampling of regularization parameter ρ
  - similar estimation performance, but (much) less efficient algorithm
- Joint Bayesian estimation for c_1 (not shown here)
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Open issues

- current model
  - estimation of GaMRF hyperparameter
  - estimation of integral scale
  - EM algorithm
- model and algorithm with other multivariate priors
  - joint estimation-segmentation in space
  - temporal change detection and estimation
  - joint estimation-segmentation in time / space
- applications: group level

Thank you for your attention

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