BOOTSTRAP TESTS FOR THE TIME CONSTANCY OF MULTIFRACTAL ATTRIBUTES

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SUMMARY On open and controversial issue in empirical data analysis is to decide whether scaling and multifractal properties observed in empirical data actually exist, or whether they are induced by intricate non stationarities. To contribute to answering this question, we propose here a non parametric bootstrap and wavelet Leaders based procedure aiming at testing the constancy along time of multifractal attributes estimated over adjacent non overlapping time windows.

SCALING (OR MULTIFRACTAL) ANALYSIS

SCALING

Scale Invariance

- $X(t),\ t \in [0,n)$ - Process under analysis
- $T_X(a,k)$ - Multiresolution quantities of X jointly depend on analysis scale a and time position t

$$\frac{1}{n_a}\sum_{k=1}^{n_a} |T_X(a,k)|^q \simeq c_q |a|^{\zeta(q)}$$

for statistical orders $q \in [q_*^-, q_*^+]$
for scales $a = 2^j \in [a_m, a_M]$, $a_M/a_m >> 1$

- $\zeta(q)$ - Scaling exponents
- $D(h)$ - Multifractal spectrum can be obtained as the Legendre transform of $\zeta(q)$

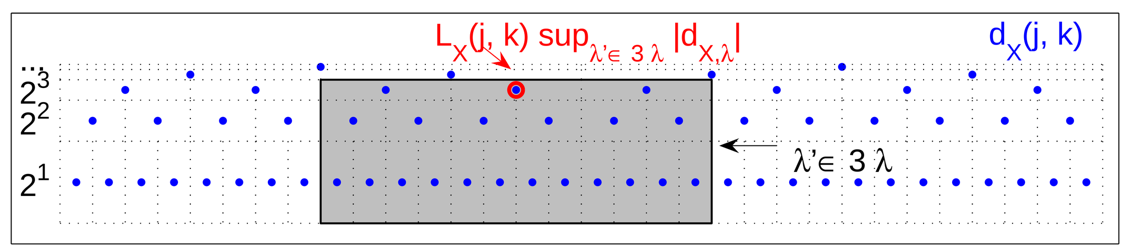
Multiresolution Quantities $T_X(a,k)$

Discrete Wavelet Coefficients (vanishing moments N_ψ)

$$d_X(j,k) = \langle \psi_{j,k} | X \rangle$$

Wavelet Leaders [*Jaffard04*]

$$L_X(j,k) = \sup_{\lambda' \in 3\lambda_{j,k}} |d_{\lambda'}|$$



ESTIMATION

Log-Cumulants [*CastaingGagneMarchand93*]

$$\zeta(q) = \sum_{p=1}^{\infty} c_p \frac{q^p}{p!}$$ polynomial expansion

- $\forall p \geq 1:$ $C(2^j,p) = c_p^0 + c_p \ln 2^j$
  $C(2^j,p)$: p-th cumulant of $\ln |T_X(2^j,\cdot)|$

Estimation Procedures

a) Calculate n_j coefficients $T_X(2^j,k)$
b) $\hat{C}(2^j,p)$ - standard estimate of p-th cumulant of $\ln |T_X(2^j,\cdot)|$
c) Linear regressions: $\hat{c}_p = (\log_2 e) \cdot \sum_{j=j_1}^{j_2} w_j \hat{C}(2^j,p)$

SCALING AND NON STATIONARITY

PROBLEM

CONTROVERSY: **Scale Invariance ↔ Non Stationary**

Do scaling actually exist in data, or are they the consequence of non stationarities that conspire to mimic scaling behavior?

Controversy

Three categories:

1. Data scale invariant + smooth trend (mean, variance) superimposed
2. Data scale invariant + non stationary variability scaling parameters
3. Data not scale invariant → strong non stationary variability confused with scaling property

Consequences

1. - Smooth trend likely to impair analysis (cf. [*VeitchAbry99*]) not further considered here

2. & 3. - Non stationary variability can correspond to many realities:
→ much more involved
→ blind analysis: misleading interpretations of scaling
→ detection of such situations of crucial practical importance

GOALS

Discrimination of true scaling against various forms of non stationary variability for multifractal processes

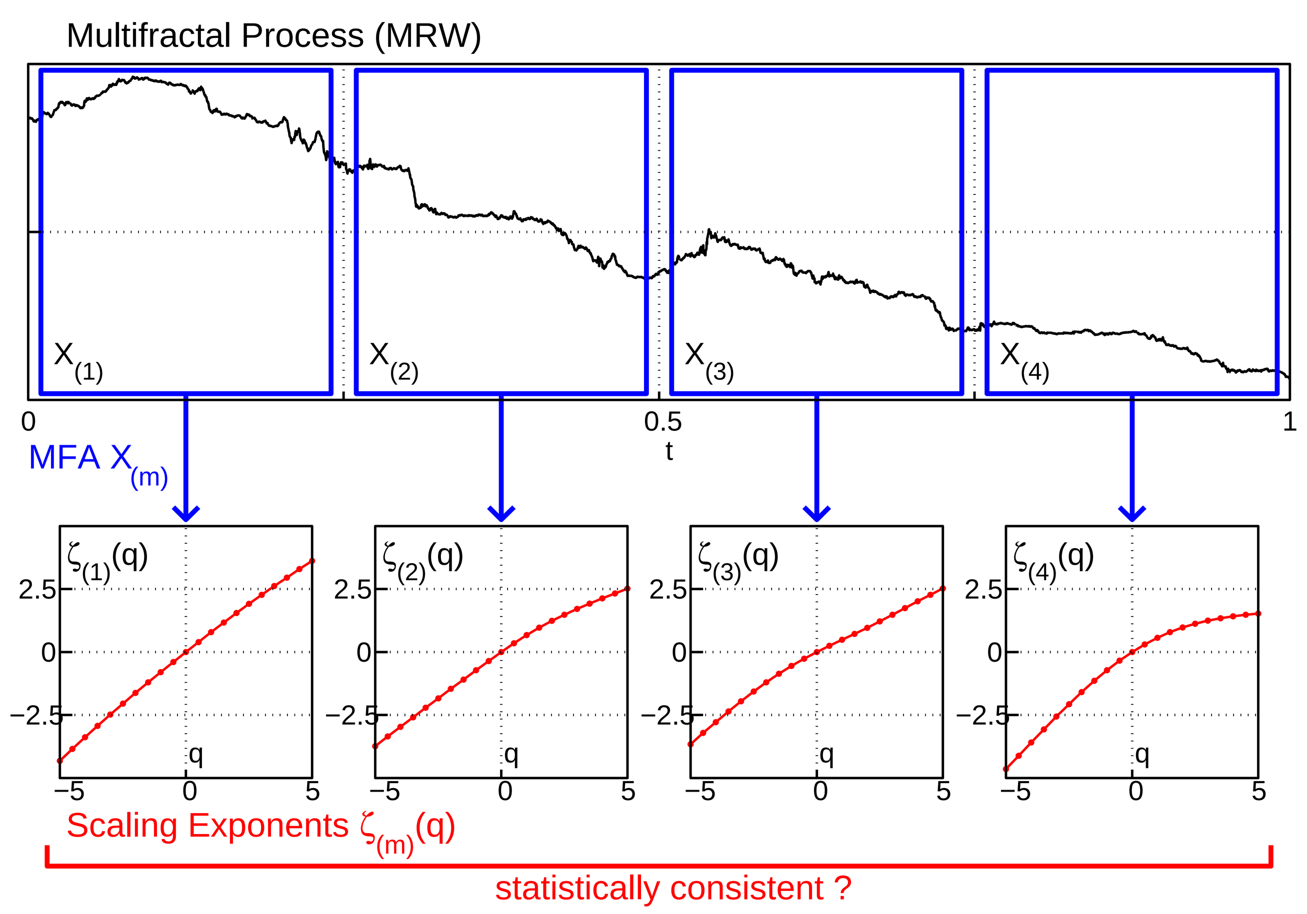
Heuristic

Estimates over non overlapping adjacent windowed time series $X_{(m)}$

- Statistically consistent ⟹ scale invariance
- Not statistically consistent ⟹ some form of non stationarity ([*VeitchAbry01*] for Gaussian H-sssi process)

Extension to multifractal processes

- Changes in methodology:
  1. Description of processes → whole collection of attributes $\zeta(q)$, c_p
  2. Wavelet Leaders based estimation → non linear transform of data
- + Additional difficulties:
  3. Strongly non Gaussian, heavy tailed, correlated processes
- Analytical approach:
  → properties of statistics underlying test ??
- Proposed approach:
  → non parametric bootstrap based test procedure



BOOTSTRAP TIME CONSTANCY TEST

Test Principle

- Multifractal attribute under test: $\theta \in \{c_p\}$
- Test identical mean for independent estimates:

$$H_0: \quad \theta_{(1)} = \theta_{(2)} = \cdots = \theta_{(M)}$$

Bootstrap Test Statistic

- M Leader based estimates $\hat{\theta}_{(m)}$ from adjacent non overlapping subsets $X_{(m)}$
- Resample from Leaders $\{L_{X_{(m)}}(j,k)\}$ corresponding to subsets $X_{(m)} \to \hat{\sigma}^{2*}_{(m)}$
- Under H_0, distribution of T_θ independent of precise means/variances of $\hat{\theta}_{(m)}$

$$T_\theta = \sum_{m=1}^{M} \frac{1}{\hat{\sigma}^{2*}_{(m)}} \left(\hat{\theta}_{(m)} - \frac{\sum_{n=1}^{M} \frac{\hat{\theta}_{(m)}}{\hat{\sigma}^{2*}_{(m)}}}{\sum_{n=1}^{M} \frac{1}{\hat{\sigma}^{2*}_{(m)}}} \right)^2$$

Bootstrap Null Distribution Estimation

1. Resample from *complete* set $\{L_X(j,k)\}$ of Leaders
2. Then cut into M subsets $\{(L_X^{*(b)})_{(m)}\}$
   → Bootstrap subset estimates $\hat{\theta}^*_{(m)}$ have same conditional distribution
   → T_θ^* always reproduces null distribution of T_θ

- $\hat{\sigma}^{2**}_{(m)}$ from double bootstrap

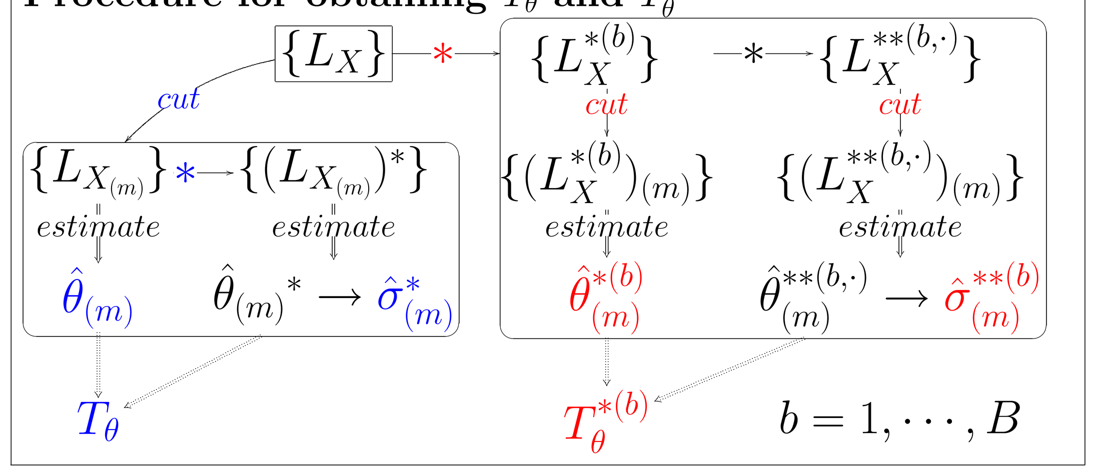
$$T_\theta^* = \sum_{m=1}^{M} \frac{1}{\hat{\sigma}^{2**}_{(m)}} \left(\hat{\theta}^*_{(m)} - \frac{\sum_{n=1}^{M} \frac{\hat{\theta}^*_{(m)}}{\hat{\sigma}^{2**}_{(m)}}}{\sum_{n=1}^{M} \frac{1}{\hat{\sigma}^{2**}_{(m)}}} \right)^2$$

Bootstrap Test

$d_\theta = 1$ if $T_\theta > T^*_{\theta,C}$ and 0 otherwise

- $T^*_{\theta,C}$ - critical value:
  $(1-\alpha)$ quantile of empirical distribution of T_θ^*

Procedure for obtaining T_θ and T_θ^*



Bootstrapping Wavelet Leaders

for $b = 1, \cdots, B$
  for $j = 1, \cdots, j_2$
    From $\{L_X(j,1), \cdots, L_X(j,n_j)\}$ random draw, with replacement, circular and overlapping blocks to form the unsorted collection $\{L_X^{*(b)}(j,1), \cdots, L_X^{*(b)}(j,n_j)\}$
  end
end

PERFORMANCE ASSESSMENT AND RESULTS

MONTE CARLO SIMULATIONS

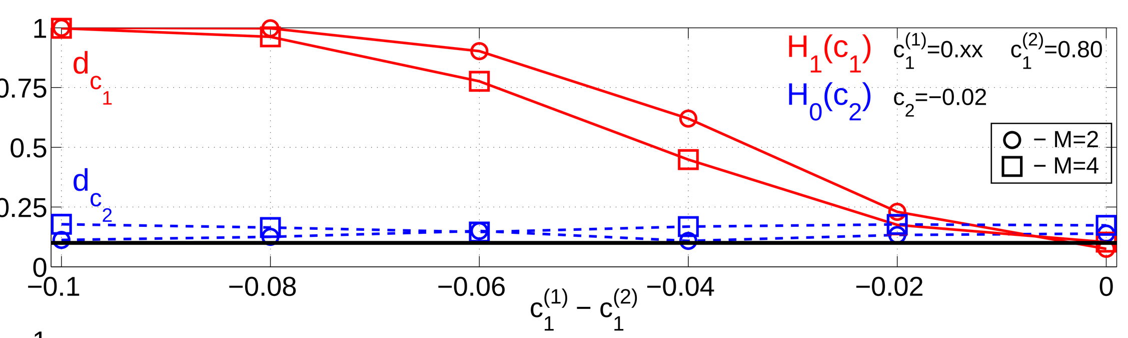
Simulation Setup

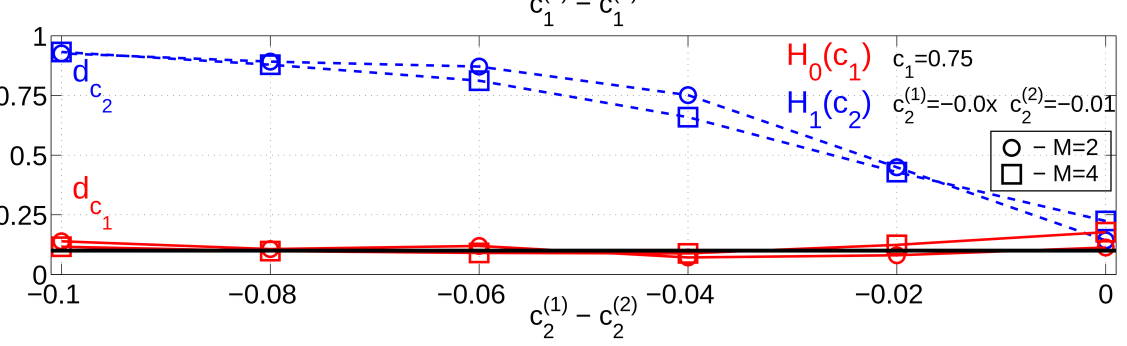
$N_{MC} = 1000$	$N = 2^{15}$	$B = B_2 = 99$	$\alpha = 0.1$	$N_\psi = 3$
$j_1 = 3$	$j_2(M) = \log_2 N - \log_2 M - (2N_\psi - 1)$			

Multifractal Random Walk (MRW) [*Mandelbrot99*]

- $c_1, c_2 \neq 0$; $p \geq 3:\ c_p \equiv 0$

Performance Assessment: $\hat{d}_\theta^{H_{(\cdot)}} = \widehat{\mathbb{E}}_{N_{MC}}\{d_\theta | H_{(\cdot)}\}$





BOOTSTRAP TEST PERFORMANCE

Test performance under H_0

- Constant multifractal attributes $\{c_1,\ c_2\}$

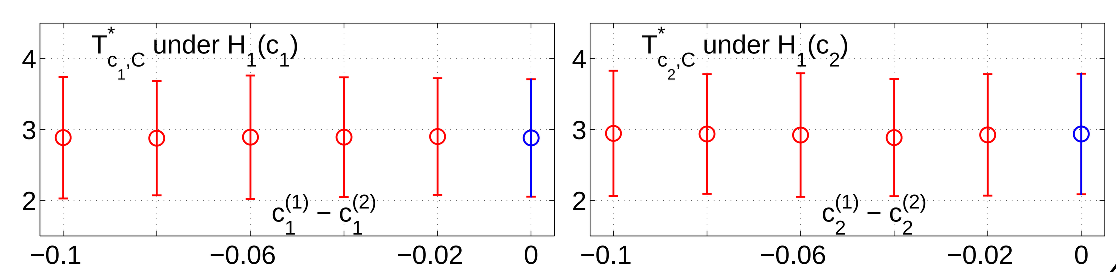
Test performance under H_1

- Simplest alternative: piecewise constant multifractal attributes
  → concatenation of two equal-length MRW with $\{c_1^{(i)}, c_2^{(i)}\}_{i=1,2}$
- $\mathbf{H_1(c_1)}$: non constant c_1: $c_1^{(1)} = \{0.70, 0.72, \cdots, 0.80\}$, $c_1^{(2)} = 0.8$
  constant c_2: $c_2^{(1)} = c_2^{(2)} = -0.02$
- $\mathbf{H_1(c_2)}$: constant c_1: $c_1^{(1)} = c_1^{(2)} = 0.75$
  non constant c_2: $c_2^{(1)} = \{-0.11, -0.10, \cdots, -0.01\}$, $c_2^{(2)} = -0.01$

NULL DISTRIBUTION ESTIMATION

Critical value $T^*_{c_p,C}$ under $H_1(c_p)$

- independent of $c_p^{(1)} - c_p^{(2)}$
- equal $T^*_{c_p,C}$ under $H_0(c_p)$



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