Pitfall in Multifractal Analysis of Negative Regularity

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Résumé – La récente définition de p-exposants et p-leaders étend l'application de l'analyse multifractale à des fonctions ou signaux de régularité négative, à conditions que ceux-ci soient localement L^p. Le formalisme multifractal, mis en oeuvre sur des signaux à temps discret, qui ne satisferaient pas cette contrainte théorique, produira toujours un résultat pratique, qui semblera réaliste, mais n'aura pas de validité théorique. Il sera cependant impossible a posteriori de s'en rendre compte. Dans ce travail, nous utilisons le modèle simple des cascades d'ondelettes déterministes pour étudier théoriquement la forme que prendra l'estimation pratique du spectre multifractal de fonctions non localement L^p. Nous conjecturons que la forme obtenue est valide en général et le validons au moyen de simulations numériques.

Abstract – The recent introduction of p-exponents and p-leaders extends the application of wavelet leader multifractal analysis to functions or signals with negative regularity. These new quantities are defined only for functions that are locally in L^p. However, in practice, estimations from discrete data can always be computed, even if the underlying function that models the data is not in L^p. In this case, the analysis is meaningless but indistinguishable from a valid one. In this contribution, we use a very simple function model provided by deterministic wavelet cascades to study the behavior of multifractal estimates when they are computed from discrete data that is not modeled by a function in L^p, and show that the result is a spectrum with correct shape but shifted so as to exactly be in the L^p limit. We also use numerical simulations on various multifractal random processes to show that the validity of our results extends beyond the simple model that we used.

1 Introduction

Multifractal analysis. Multifractal analysis is nowadays a relevant and valid signal processing tool. It has been successfully used to analyze, describe, model and classify the dynamics of signals in numerous applications, which include hydrodynamic turbulence [11], biomedical data [4] and internet traffic [3], among many others. Multifractal analysis describes a function $X(t)$ based on its *local regularity*, commonly measured by the *Hölder exponent* $h(t)$ [6]. It is, however, often the case in practice that data to be analyzed contain negative regularity. In that case, the Hölder exponent can not be used since it is a strictly positive quantity. To overcome this limitation, a generalization has recently been proposed, the *p-exponent* $h_p(t)$ [1, 8, 9], which can take negative values, $h_p(t) \geq -1/p$, for fixed $p \in (1, +\infty]$, and can be computed for the less restrictive set of functions which belong to L^p. This new setting contains the Hölder exponent as the limit case $p = +\infty$.

Multifractal formalism. Rather than characterizing $X(t)$ via the function $h_p(t)$, multifractal analysis provides a global and geometric description of the fluctuations of the values of $h_p(t)$ along time : the *multifractal spectrum* $D_p(h)$, defined as the Hausdorff dimension of the set of points where $h_p(t) = h$. This theoretical definition is, however, not constructive for obtaining procedures for computing $D_p(h)$ from finite resolution data. Instead, practical estimation is achieved through a procedure referred to as the *multifractal formalism*. It relies on the use of relevant quantities for measuring p-exponents, referred to as *p-leaders* [1,8,9]. For $p = +\infty$ (Hölder exponents), one recovers the well-known wavelet leaders [6].

Practical pitfalls of negative regularity. The multifractal formalism requires the parameter p to be fixed a priori. The precise choice of p is critical because for a function that is not locally in L^p, the theoretical quantity on which the analysis is based (the p-exponent) and the multiresolution quantities used for its estimation (the p-leaders) are theoretically ill defined and take infinite values. However, in practice, estimation is performed on discrete data, i.e., on a finite-valued finite-resolution sampled version of a function. Thus, the multiresolution quantities can always be computed, since they consist of finite sums of finite values, and are hence finite for any (positive) value of p. The multifractal formalism therefore always provides finite-valued estimates for the multifractal spectrum obtained from discrete data. Yet, if the function modeling the data does not belong to L^p, the result of the analysis is meaningless, since the underlying quantities and the multifractal spectrum are not theoretically defined, and misleading, since the estimated multifractal spectrum is nevertheless indistinguishable from a valid spectrum of a function that is in L^p.

Goals and contributions. The goal of the present contribution is to shed light on this important pitfall and to study the multifractal spectrum obtained with the p-leader multifractal formalism when the condition $X \in L^p$ is violated. To that end, we study theoretically the multifractal spectrum obtained for finite resolution data from a simple deterministic multifractal model. We obtain explicit expressions of the estimated spectra when the model function is not in L^p. We provide numerical simulations for different synthetic multifractal random processes that indicate that this theoretical result is a valid approximation in general for multifractal processes, beyond the simple deterministic model it is based on.

2 p-leaders multifractal analysis

p-exponent regularity. Let $X \in L^p_{loc}(\mathbb{R})$ for $p \geq 1$. X is said to belong to $T^p_\alpha(t)$, with $\alpha > -1/p$, if there exist $C, R > 0$ and a polynomial P_t (with $deg(P_t) \leq \alpha$) such that $\forall a < R$, $\left(\frac{1}{a}\int_{t-a/2}^{t+a/2} |X(u) - P_t(u-t)|^p \, du\right)^{1/p} \leq C a^\alpha$. The *p-exponent* of X at t is defined as $h_p(t) = \sup\{\alpha \; : \; X \in T^p_\alpha(t)\}$. It is a natural substitute for the Hölder exponent when dealing with functions which are not bounded (but belong to L^p) and admits negative values $h_p > -1/p$. The Hölder exponent is recovered for $p = +\infty$. The p-exponents can be measured using p-leaders, defined in the next paragraph.

Wavelet coefficients and p-leaders. Let $\{X(t)\}_{t\in\mathbb{R}}$ denote the signal to be analyzed. Let ψ denote the mother wavelet, characterized by its number of vanishing moments N_ψ, a strictly positive integer such that $\int_{\mathbb{R}} t^k \psi(t) dt = 0, \forall k = 0, \ldots, N_\psi - 1$, and $\int_{\mathbb{R}} t^{N_\psi} \psi(t) dt \neq 0$. Let $\{\psi_{j,k}(t) = 2^j \psi(2^j t - k)\}_{(j,k)\in\mathbb{N}^2}$ be the orthonormal basis of $L^2(\mathbb{R})$ formed by dilations and translations of ψ. The (L^1-normalized) discrete wavelet transform coefficients are defined as $c_{j,k} = 2^{j/2}\langle\psi_{j,k}|X\rangle$ (cf., e.g., [10], for more details on wavelet transforms).

Now let $\lambda = \lambda_{j,k} = [k2^j, (k+1)2^j)$ denote a dyadic interval and $3\lambda = \bigcup_{m\in\{-1,0,1\}} \lambda_{j,k+m}$ the union with its two neighbours. The *p-leaders* are defined for $X \in L^p$ as [1,8,9]

$$\ell^{(p)}_{j,k} \triangleq \Big(\sum_{\lambda' \subset 3\lambda} |c_{\lambda'}|^p \, 2^{j-j'} \Big)^{\frac{1}{p}}, \tag{1}$$

where the sum involves all the wavelet coefficients in a narrow time neighbourhood of $t = 2^{-j}k$ for all finer scales $j' \geq j$. It can be shown that they reproduce p-exponents in the limit of fine scales $j \to \infty$ as $\ell^{(p)}_{j,k} \sim 2^{-jh_p(2^j k)}$ [1,8,9].

The classical wavelet leaders are given for $p = +\infty$, in which case (1) reduces to $\ell^{(\infty)}_{j,k} \triangleq \sup_{\lambda'\subset 3\lambda} |c_{\lambda'}|$.

Multifractal formalism. The p-leader multifractal formalism for computing $D_p(h)$ is defined as follows. First, the structure functions are computed, defined as

$$S_p(j,q) = 2^{-j} \sum_{k=1}^{2^j} \left(\ell^{(p)}_{j,k}\right)^q \sim 2^{-j\zeta_p(q)}, j \to \infty. \tag{2}$$

The *scaling function* $\zeta_p(q)$ is estimated by means of linear regressions of $\log_2 S_p(j,q)$ versus j. It can be shown that $\zeta_p(q)$ is the Legendre transform of $D_p(h)$, and thus the concave hull of the multifractal spectrum can be recovered from $\zeta_p(q)$ as [6]

$$D_p(h) \leq \mathcal{L}_p(h) \triangleq \min_q (1 + qh - \zeta_p(q)). \tag{3}$$

In practice, the function $\mathcal{L}_p(h)$ is the only accessible quantity and is used as the estimate of $D_p(h)$.

Function space requirements. p-exponents and p-leaders are theoretically defined only for functions $X \in L^p$. This requirement can be checked based on the *wavelet scaling function* $\eta(p)$, practically defined by the relation

$$2^{-j} \sum_{k=1}^{2^j} |c_{j,k}|^p \sim 2^{-j\eta(p)}, \quad j \to \infty, \; p \geq 0. \tag{4}$$

It can be shown that if $\eta(p) > 0$, then $X \in L^p$, and that the condition $\eta(p) > 0$ implies that the multifractal spectrum must satisfy $D_p(h) \leq 1 + ph$ [1,8].

Finally, note that the condition $X \in L^p$ is much more restrictive for wavelet leaders ($p = +\infty$) since $L^\infty \subseteq L^p$ for all $p \geq 1$.

3 Analysis of Legendre spectra limits

We study the behavior of the multifractal spectrum (3) that is obtained from finite resolution data for functions $X \in L^p$ and $X \notin L^p$. We base our analysis on simple multifractal model functions provided by (binomial) *deterministic wavelet cascades* (DWC), defined as follows [11] : Let $0 < \omega_0 < \omega_1$ and let the parent coefficient at scale $j = 0$ equal 1, $c_{0,1} = 1$. At scale $j > 0$, the 2^j wavelet coefficients are obtained as $c_{j,2k} = \omega_0 c_{j-1,k}$ and $c_{j,2k+1} = \omega_1 c_{j-1,k}$, and therefore take values $c_{j,k} \in \{\omega_0^n \omega_1^{j-n}, \; n = 0, \ldots, j\}$. The corresponding function is obtained by an inverse wavelet transform.

Substitution of the coefficients $c_{j,k}$ in (4) yields

$$2^{-j} \sum_{k=1}^{2^j} (c_{j,k})^q = 2^{-j}(\omega_0^q + \omega_1^q)^j = 2^{-j\eta(q)}$$

from which we identify the wavelet scaling function of DWC

$$\eta(q) = 1 - \log_2(\omega_0^q + \omega_1^q). \tag{5}$$

The Legendre spectrum of DWC reads

$$\mathcal{L}^{\omega_0,\omega_1}_\eta(h) = \min_q (1 + qh - \eta(q)). \tag{6}$$

It can be easily seen in (5) that the cascade is in L^p if $\omega_0^p + \omega_1^p < 2$ (and hence in L^∞ as long as $\omega_1 < 1$).

Restricted p-leaders analysis. For simplicity, we consider the restricted p-leaders defined by $\ell^{(p)}_\lambda = \left(\sum_{\lambda'\subset\lambda} |c_{\lambda'}|^p 2^{j-j'}\right)^{\frac{1}{p}}$. It was shown that structure functions with $\ell^{(p)}_\lambda$ yield quantities equivalent to (2) so that the corresponding scaling functions (defined in the limit of fine scales) coincide [7]. We suppose that the DWC is available at finite resolution and that the largest available scale is J. Let $p \geq 0$. Using the change of variables $l = j' - j$ and the multiplicative structure of the cascade we have

$$\ell^{(p)}_\lambda = c_\lambda \Big(\sum_{l=0}^{J-j} (\omega_0^p + \omega_1^p)^l 2^{-l} \Big)^{\frac{1}{p}} = c_\lambda \Big(\sum_{l=0}^{J-j} 2^{-\eta(p)l} \Big)^{\frac{1}{p}}.$$

In the limit of infinite resolution $J \to \infty$, or coarse scales $j \to -\infty$, the sum diverges if $\eta(p) < 0$. Yet, for finite $J - j$,

$$\ell_\lambda^{(p)} = c_\lambda \left(\frac{1 - 2^{-(J-j+1)\eta(p)}}{1 - 2^{-\eta(p)}} \right)^{1/p}.$$

The p-leader structure function is therefore given by

$$\begin{aligned} S_p(j,q) &= 2^{-j} \sum_{k=1}^{2^j} (c_{j,k})^q \Big(\sum_{l=0}^{J-j} 2^{-\eta(p)l} \Big)^{\frac{q}{p}} \\ &= \left(\frac{\omega_0^q + \omega_1^q}{2} \right)^j \left(\frac{1 - \left(\frac{\omega_0^p + \omega_1^p}{2}\right)^{J-j+1}}{1 - \left(\frac{\omega_0^p + \omega_1^p}{2}\right)} \right)^{\frac{q}{p}}. \quad (7) \end{aligned}$$

Cascade in L^p. The evolution of $S_p(j,q)$ with j is different from the one in (2) by a term that converges to a constant for coarse scales $j \to -\infty$ when $\omega_0^p + \omega_1^p < 2$. As a result, the power law scaling defining $\zeta_p(q)$ in (2) can only be measured at coarse scales. In the limit of coarse scales,

$$S_p(j,q) \overset{j \to -\infty}{\sim} 2^{-j\eta(q)} \left(1 - \frac{\omega_0^p + \omega_1^p}{2} \right)^{-\frac{q}{p}},$$

and hence $\zeta_p(q) = \eta(q)$ with $\eta(q)$ defined in (5). Therefore, $\mathcal{L}_p(h) \equiv \mathcal{L}_\eta^{\omega_0,\omega_1}(h)$.

Cascade not in L^p. In case the cascade is not in L^p (i.e., $\omega_0^p + \omega_1^p \geq 2$), the term $\left(\frac{\omega_0^p + \omega_1^p}{2}\right)^{J-j+1}$ in (7) diverges as a power law when $j \to -\infty$. To proceed with the analysis, we use the substitution $\omega_0 = \alpha v_0$ and $\omega_1 = \alpha v_1$ with $v_0^p + v_1^p = 2$, implying $\alpha > 1$. Then, (7) becomes

$$\begin{aligned} S_p(j,q) &= \left(\alpha^q \frac{v_0^q + v_1^q}{2} \right)^j \left(\frac{1 - (\alpha^p)^{J-j+1}}{1 - \alpha^p} \right)^{\frac{q}{p}} \\ &= \left(\frac{v_0^q + v_1^q}{2} \right)^j \left(\frac{(\alpha^p)^j - (\alpha^p)^{J+1}}{1 - \alpha^p} \right)^{\frac{q}{p}}, \quad (8) \end{aligned}$$

where the second term of the right hand side behaves as a constant when $j \to -\infty$:

$$S_p(j,q) \overset{j \to -\infty}{\sim} 2^{-j\zeta_p(q)} \left(\frac{1 - (\alpha^p)^{J+1}}{1 - (\alpha^p)} \right)^{\frac{q}{p}}$$

with $\zeta_p(q) = 1 - \log_2(v_0^q + v_1^q)$. Consequently, the multifractal spectrum does not equal $\mathcal{L}_\eta^{\omega_0,\omega_1}(h)$ but is given by

$$\mathcal{L}_p(h) \equiv \mathcal{L}_\eta^{v_0,v_1}(h), \qquad v_0^p + v_1^p = 2 \quad (9)$$

i.e., the spectrum of a cascade with multipliers v_0 and v_1 that is in L^p but not in $L^{p'}$ for any $p' > p$ (since $v_0^p + v_1^p = 2$).

Conclusion. For a DWC not in L^p, the estimated $\mathcal{L}_p(h)$ is hence shifted to the right to touch the L^p border $1 + ph$ in one single point, but does not undergo any shape deformation : It thus resembles the spectrum that would be obtained for a function X that is just at the limit of L^p, i.e., $X \in L^p$ but $X \notin L^{p'}$ for any $p' > p$. This behavior is illustrated in Fig. 1.

Conjecture. This leads us to formulate the following conjecture : When applied to any function not in L^p, the estimated Legendre spectrum $\mathcal{L}_p(h)$ is shifted to the right to touch the L^p border $1 + ph$, with no shape deformation and thus corresponds to the theoretical spectrum of an equivalent process that satisfies the L^p-constraint.

Wavelet leaders. Results for wavelet leaders can be obtained from the calculations for p-leaders in the limit $p \to \infty$. In this case, when $X \notin L^\infty$, the multifractal spectrum is given by

$$\mathcal{L}_\infty(h) = \mathcal{L}_\eta^{v_0,v_1}(h), \quad v_0 = \omega_0/\omega_1 < 1, \ v_1 = 1, \quad (10)$$

It is hence shifted to the right such that its left-most point is at $h = 0$. This behavior is illustrated in Fig. 1.

4 Numerical simulations

In this section, we provide numerical evidence for the fact that expressions (9) and (10) for the limits of Legendre spectra are generically valid for multifractal processes with negative regularity. To this end, we apply the multifractal formalism to $N_{MC} = 50$ independent realizations of sample size $N = 2^{18}$ of several random processes X with known and controlled multifractal properties. We use fractional differentiation of order ν to control the function space embedding, $X^{(\nu)} = \mathcal{F}^{-\nu}\big((\imath\omega)^\nu \mathcal{F}[X]\big)$, where $\mathcal{F}$ stands for the discrete Fourier transform, cf. [13] for details. Analysis is performed using $p = 2$ (p-leaders) and $p = \infty$ (wavelet leaders), for DWC with $(\omega_0, \omega_1) = (0.3, 1.6)$ and $\nu = 0$ (for which $X^{(\nu)} \in L^p$ with $p \leq 1.14$), and for the following multifractal random processes.

Random wavelet cascades (RWC) are built from wavelet coefficients in a similar fashion to DWC, but replacing the deterministic multipliers ω_0 and ω_1 with random variables [2]. Let $c_{j=0,k=1} = 1$. At scale $j > 0$, wavelet coefficients are obtained as $c_{j,2k} = W_{j,k}^l c_{j-1,k}$ and $c_{j,2k+1} = W_{j,k}^r c_{j-1,k}$, where $W_{j,k}^l$ and $W_{j,k}^r$ are iid positive random variables. We use log-normal multipliers W with mean μ and variance σ^2. The multifractal spectrum is given by $D(h) = 1 - (h - \mu + \nu)^2/(2\log(2)\sigma^2)$ [2]. Here, we use $\mu = 0.56$, $\sigma = 0.3$ and $\nu = 0.8$, for which RWC are not in L^p for any p.

α-stable Lévy processes $S_\alpha(t)$ are built from a symmetric α-stable measure $M(ds)$ as $S_\alpha(t) = \int_{\mathbb{R}} f(t,s) M(ds)$, where $f(t,s) = \mathbb{1}(t - s > 0) - \mathbb{1}(-s > 0)$ [12]. Its multifractal spectrum is given by $D(h) = \alpha(h + \nu)$ when $-\nu \leq h \leq 1/\alpha - \nu$, and $D(h) = -\infty$ otherwise [5]. We set $\alpha = 1.25$ and $\nu = 0.7$ (for which S_α are in L^p with $p \leq 10/7$).

Conjecture. Assuming the conjecture above is true, the estimated spectra for RWC should read $\mathcal{L}_p(h) = 1 - (h - \mu + \nu')^2/(2\log(2)\sigma^2)$ with $\nu' = \mu - \sqrt{2\log(2)\sigma^2}$ if $p = \infty$, or $\nu' = \mu - \log(2)\sigma^2$ if $p = 2$. Similarly, for $S_\alpha(t)$ it should read $\mathcal{L}_p(h) = \alpha(h + \nu')$ with $\nu' = -1/p$.

Estimation of multifractal spectra. Fig. 1 plots the p-leader (left column) and wavelet leader (right column) based estimates of $D(h)$ (black solid lines, points) for DWC (top row), RWC (center row) and S_α (bottom row), together with the theoretical spectra $\mathcal{L}_\eta^{\omega_0,\omega_1}(h)$ (red solid lines), the limit spectra (9) and

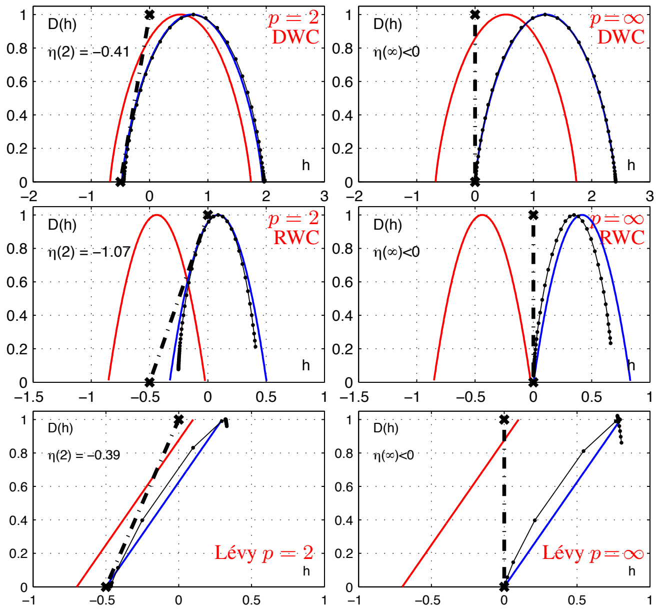

FIGURE 1 – Multifractal analysis of DWC (top row), RWC (center row) and Lévy process (bottom row) with negative regularity using p-leaders ($p = 2$, left column) and wavelet leaders ($p = \infty$, right column) : theoretical multifractal spectra $\mathcal{L}_\eta^{\omega_0,\omega_1}(h)$ (red solid lines), Legendre limit spectra $\mathcal{L}_\eta^{v_0,v_1}(h)$ (blue solid lines), estimates of multifractal spectra (black solid lines, points) and L^p limit (black dashed-dotted lines, crosses).

(10) (blue solid lines) and the L^p limits $D(h) \le 1 + ph$ (black dashed-dotted lines). First, the multifractal formalism clearly provides estimates that conspire to resemble valid multifractal spectra, despite the fact that p-exponents and p-leaders are theoretically undefined and infinite. The estimated spectra are shifted to the right with respect to the theoretical spectra, so that they are right below the L^p limit for the value of p used in the analysis, yet they precisely conserve the *shape* of the theoretical spectra so that they are *a posteriori* indistinguishable from valid spectra. Second, Fig. 1 indicates that the proposed expressions (9) and (10) for the limit spectra provide excellent models for the estimated multifractal spectra not only for DWC, but also for the synthetic random processes RWC and S_α. This is in particular remarkable for S_α, whose construction is not based on a multiplicative cascade but on an additive mechanism, which indicates the general validity of the model for processes with negative regularity. Finally, note that while inspection of the estimated multifractal spectra does not provide any indication for the fact that the L^p assumption is violated, estimates of the wavelet scaling function $\eta(p)$ are found to be negative for $p = 2$ and $p = \infty$ for all processes and hence enable to detect that the assumption $X \in L^p$ is violated and the analysis is invalid. This underlines the importance of performing a preliminary wavelet based analysis and checking the condition $\eta(p) > 0$ before applying the p-leaders multifractal formalism to data.

5 Conclusions

In this contribution, we provided a theoretical analysis of the p-leader multifractal formalism when applied to finite resolution data coming from functions with negative regularity, which are not in L^p. We used a simple deterministic model to derive expressions for the estimated multifractal spectra. The model predicts that the estimated spectra precisely resemble the theoretical multifractal spectra of functions that would be obtained by increasing the regularity of the original functions such that they are in L^p but not in $L^{p'}$ for any $p' > p$. Numerical simulations for synthetic multifractal random processes indicate the validity of this model for general processes with negative regularity. The result provides a better understanding of the p-leader and wavelet leader multifractal formalisms for discrete data and stresses the importance of checking *a priori* that data are in L^p by performing a preliminary wavelet based analysis.

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