EXAM ESTIMATION - DETECTION

21/10/2016

SOLUTIONS

**Exercise 1: Estimation**

1. Maximum likelihood estimation

(a)

$$
\begin{aligned}
f(x_1,\cdots,x_n;\beta,\theta) &= \frac{\prod_{i=1}^n x_i^{\beta-1}}{\Gamma(\beta)^n\theta^{n\beta}}\exp\left(-\frac{\sum_{i=1}^n x_i}{\theta}\right)\\
\ln f(x_1,\cdots,x_n;\beta,\theta) &= -n\beta\ln\theta - n\ln\Gamma(\beta) + (\beta-1)\sum_{i=1}^n \ln x_i - \frac{\sum_{i=1}^n x_i}{\theta}\\
\frac{\partial \ln f(x_1,\cdots,x_n;\beta,\theta)}{\partial\theta} &= -\frac{n\beta}{\theta} + \frac{\sum_{i=1}^n x_i}{\theta^2} \stackrel{!}{=} 0 \qquad \Longrightarrow \theta = \frac{\sum_{i=1}^n x_i}{n\beta} = \frac{\bar{x}}{\beta}\\
\frac{\partial^2 \ln f(x_1,\cdots,x_n;\beta,\theta)}{\partial\theta^2} &= \frac{n\beta}{\theta^2} - 2\frac{\sum_{i=1}^n x_i}{\theta^3}\\
\left.\frac{\partial^2 \ln f(x_1,\cdots,x_n;\beta,\theta)}{\partial\theta^2}\right|_{\theta=\bar{x}/\beta} &= \frac{n\beta}{\bar{x}^2/\beta^2} - 2\frac{n\bar{x}}{\bar{x}^3/\beta^3} = -\frac{n\beta^3}{\bar{x}^2} < 0\\
&\Longrightarrow \hat{\theta}_{ML} = \frac{\sum_{i=1}^n x_i}{n\beta} = \frac{\bar{x}}{\beta}
\end{aligned}
$$

(b)

$$
\begin{aligned}
\mathbb{E}\left[\hat{\theta}_{ML}\right] &= \mathbb{E}\left[\frac{\sum_{i=1}^n x_i}{n\beta}\right] = \frac{\sum_{i=1}^n \mathbb{E}\left[x_i\right]}{n\beta} = \frac{n\beta\theta}{n\beta} = \theta \qquad \Longrightarrow \text{unbiased}\\
\text{Var}\left[\hat{\theta}_{ML}\right] &= \text{Var}\left[\frac{\sum_{i=1}^n x_i}{n\beta}\right] = \frac{\sum_{i=1}^n \text{Var}\left[x_i\right]}{n^2\beta^2} = \frac{n\beta\theta^2}{n^2\beta^2} = \frac{\theta^2}{n\beta} \qquad \Longrightarrow \text{convergent}
\end{aligned}
$$

(c)

$$
\begin{aligned}
I(\theta) &= \mathbb{E}\left[-\frac{\partial^2 \ln f(x_1,\cdots,x_n;\beta,\theta)}{\partial\theta^2}\right] = -\frac{n\beta}{\theta^2} + 2\frac{\sum_{i=1}^n \mathbb{E}[x_i]}{\theta^3} = \frac{n\beta}{\theta^2}\\
CRB(\theta) &= \frac{\theta^2}{n\beta} \qquad = \text{Var}\left[\hat{\theta}_{ML}\right] \;\Longrightarrow\; \hat{\theta}_{ML} \text{ is the efficient estimator for } \theta
\end{aligned}
$$

(d) Using the characteristic function of the Gamma law with parameters $a$ and $b$, we have

$$
\begin{aligned}
\varphi_X(t) &= \mathbb{E}\left[\mathrm{e}^{itX}\right] = (1-itb)^{-a}\\
\varphi_{Y=\sum_{i=1}^n X_i}(t) &= \mathbb{E}\left[\mathrm{e}^{it\sum_{i=1}^n X_i}\right] = \prod_{i=1}^n \mathbb{E}\left[\mathrm{e}^{itX_i}\right] = \prod_{i=1}^n \varphi_X(t) = (1-itb)^{-na}
\end{aligned}
$$

and therefore, $Y=\sum_{i=1}^n X_i \sim \mathcal{G}(na,b)$. The probability distribution of $Z=cX$ is

$$
g_Z(z) = f_X(x(z))\left|\frac{dx}{dz}\right| = \frac{1}{\Gamma(a)b^a}\left(\frac{y}{c}\right)^{a-1}\exp\left(-\frac{y}{bc}\right)\frac{1}{a} = \frac{1}{\Gamma(a)(bc)^a}y^{a-1}\exp\left(-\frac{y}{bc}\right)
$$

and hence $Z=cX\sim\mathcal{G}(a,bc)$. Substituting $a=\beta$, $b=\theta$ and $c=\frac{1}{n\beta}$ gives

$$
\hat{\theta}_{ML} = \frac{\sum_{i=1}^n x_i}{n\beta} \;\sim\; \mathcal{G}\left(n\beta, \frac{\theta}{n\beta}\right).
$$

2. Method of moments

   (a)

$$\mathbb{E}[X] = \beta\theta \;\longrightarrow\; \hat{\theta}_{m_1} = \frac{\bar{x}}{\beta}$$

   Hence $\hat{\theta}_{m_1} = \hat{\theta}_{ML}$ and $\hat{\theta}_{m_1}$ it is unbiased, convergent, and the efficient estimator for $\theta$.

   (b)

$$\mathbb{E}[X^2] = \text{Var}[X] + \mathbb{E}[X]^2 = \beta\theta^2 + \beta^2\theta^2 = \theta^2\beta(1+\beta)$$
$$\theta = \sqrt{\frac{\mathbb{E}[X^2]}{\beta(1+\beta)}} \;\longrightarrow\; \hat{\theta}_{m_2} = \sqrt{\frac{\sum_{i=1}^n x_i^2}{n\beta(1+\beta)}}$$

   (c) With $n = 1$, the estimator simplifies to $\hat{\theta}_{m_2} = \frac{x}{\sqrt{\beta(1+\beta)}}$.

$$b_{m_2}(\theta) = \mathbb{E}[\hat{\theta}_{m_2}] - \theta = \frac{\beta\theta}{\sqrt{\beta(1+\beta)}} - \theta = \theta\left(\sqrt{\frac{\beta}{1+\beta}} - 1\right) = \theta\frac{\sqrt{\beta} - \sqrt{1+\beta}}{\sqrt{1+\beta}}$$
$$\text{Var}[\hat{\theta}_{m_2}] = \frac{\beta\theta^2}{\beta(1+\beta)} = \frac{\theta^2}{1+\beta}$$
$$mse_{m_2}(\theta) = b_{m_2}(\theta)^2 + \text{Var}[\hat{\theta}_{m_2}] = \theta^2\left(\sqrt{\frac{\beta}{1+\beta}} - 1\right)^2 + \frac{\theta^2}{1+\beta} = 2\theta^2\left(1 - \sqrt{\frac{\beta}{1+\beta}}\right)$$

3. Bayesian estimation with Jeffrey's prior for $\theta$

   (a) From 1.(c), we immediately have $p(\theta) \propto \sqrt{I(\theta)} = \sqrt{\frac{n\beta}{\theta^2}} = \frac{\sqrt{n\beta}}{\theta}$.

   (b)

$$f(\theta; x, \beta) \propto \frac{\prod_{i=1}^n x_i^{\beta-1}}{\Gamma(\beta)^n\theta^{n\beta}} \exp\left(-\frac{\sum_{i=1}^n x_i}{\theta}\right) \frac{\sqrt{n\beta}}{\theta}$$
$$\ln f(\theta; x, \beta) \propto -(n\beta+1)\ln\theta - n\ln\Gamma(\beta) + (\beta-1)\sum_{i=1}^n \ln x_i - \frac{\sum_{i=1}^n x_i}{\theta} + \frac{1}{2}\ln(n\beta)$$
$$\frac{\partial \ln f(\theta; x, \beta)}{\partial\theta} = -\frac{n\beta+1}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^n x_i \overset{!}{=} 0 \implies \theta = \frac{\sum_{i=1}^n x_i}{n\beta+1}$$
$$\left.\frac{\partial^2 \ln f(\theta; x, \beta)}{\partial\theta^2}\right|_{\theta=\frac{\sum_{i=1}^n x_i}{n\beta+1}} = \frac{n\beta+1}{\theta^2} - \frac{2}{\theta^3}\left.\sum_{i=1}^n x_i\right|_{\theta=\frac{\sum_{i=1}^n x_i}{n\beta+1}} = -\frac{(n\beta+1)^3}{(\sum_{i=1}^n x_i)^2} < 0$$
$$\implies \hat{\theta}^J_{MAP} = \frac{\sum_{i=1}^n x_i}{n\beta+1}$$
$$\hat{\theta}^J_{MAP} = \sum_{i=1}^n x_i \frac{n\beta}{n\beta(n\beta+1)} = \sum_{i=1}^n x_i \frac{n\beta+1\;-1}{n\beta(n\beta+1)} = \hat{\theta}_{ML} - \frac{\sum_{i=1}^n x_i}{n\beta(n\beta+1)}$$

4. Bayesian estimation with inverse Gamma prior $\mathcal{IG}(k,\tau)$ for $\theta$: $\quad \theta \sim \mathcal{IG}(k,\tau)$

(a)

$$
\begin{aligned}
f(x_1,\cdots,x_n;\beta,\theta) &= \frac{\prod_{i=1}^n x_i^{\beta-1}}{\Gamma(\beta)^n\theta^{n\beta}} \exp\left(-\frac{\sum_{i=1}^n x_i}{\theta}\right) \\
p_\theta(\theta;k,\tau) &= \frac{\tau^k}{\Gamma(k)}\theta^{-k-1}\exp\left(-\tau/\theta\right) \\
f(\theta;x,\beta,k,\tau) &= \frac{\prod_{i=1}^n x_i^{\beta-1}}{\Gamma(\beta)^n\theta^{n\beta}} \exp\left(-\frac{\sum_{i=1}^n x_i}{\theta}\right)\frac{\tau^k}{\Gamma(k)}\theta^{-k-1}\exp\left(-\tau/\theta\right) \\
&= \underbrace{\frac{\tau^k\prod_{i=1}^n x_i^{\beta-1}}{\Gamma(k)\Gamma(\beta)^n}}_{C}\theta^{-(n\beta+k)-1}\exp\left(-\frac{\sum_{i=1}^n x_i+\tau}{\theta}\right) \\
&= \mathcal{IG}\left(n\beta+k,\sum_{i=1}^n x_i+\tau\right)
\end{aligned}
$$

(b)

$$
\begin{aligned}
\ln f(\theta;x,\beta,k,\tau) &= \ln C-(n\beta+k+1)\ln\theta-\frac{\sum_{i=1}^n x_i+\tau}{\theta} \\
\frac{\partial \ln f(\theta;x,\beta,k,\tau)}{\partial\theta} &= -\frac{n\beta+k+1}{\theta}+\frac{\sum_{i=1}^n x_i+\tau}{\theta^2} \quad \stackrel{!}{=} 0 \\
\hat{\theta}_{MAP} &= \frac{\sum_{i=1}^n x_i+\tau}{n\beta+k+1}
\end{aligned}
$$

(One verifies that $\frac{\partial^2 \ln f(\theta;x,\beta,k,\tau)}{\partial\theta^2}\Big|_{\theta=\frac{\sum_{i=1}^n x_i+\tau}{n\beta+k+1}} < 0$.)

(c) For $X\sim\mathcal{IG}(k,\tau)$ we have

$$
\begin{aligned}
\mathbb{E}[X] &= \int x\frac{\tau^k}{\Gamma(k)}x^{-k-1}\exp\left(-\tau/x\right)dx = \int\frac{\tau^k}{\Gamma(k)}x^{-(k-1)-1}\exp\left(-\tau/x\right)dx \\
&= \tau\frac{\Gamma(k-1)}{\Gamma(k)}\underbrace{\int\frac{\tau^{k-1}}{\Gamma(k-1)}x^{-(k-1)-1}\exp\left(-\tau/x\right)dx}_{=\int\mathcal{IG}(k-1,\tau)\,=\,1} = \tau\frac{\Gamma(k-1)}{\Gamma(k)} = \frac{\tau}{k-1}.
\end{aligned}
$$

Since the posterior law of $\theta$ is $\mathcal{IG}\left(n\beta+k,\sum_{i=1}^n x_i+\tau\right)$, we immediately have

$$
\hat{\theta}_{MMSE} = \frac{\sum_{i=1}^n x_i+\tau}{n\beta+k+1}.
$$

**Exercise 2: Detection**

1. We want to test the hypotheses $H_0 : \theta = \theta_0$ and $H_1 : \theta = \theta_1$ with $\theta_1 > \theta_0$.

   (a)

$$\begin{aligned}
\text{reject } H_0 \text{ if} \quad & \frac{f(x_1, \cdots, x_n; \beta, \theta | H_1)}{f(x_1, \cdots, x_n; \beta, \theta | H_0)} > K_\alpha \\
& \frac{\prod_{i=1}^n x_i^{\beta-1} \Gamma(\beta)^n \theta_0^{n\beta} \exp\left(-\frac{\sum_{i=1}^n x_i}{\theta_1}\right)}{\prod_{i=1}^n x_i^{\beta-1} \Gamma(\beta)^n \theta_1^{n\beta} \exp\left(-\frac{\sum_{i=1}^n x_i}{\theta_0}\right)} > K_\alpha \\
& \left(\frac{\theta_0}{\theta_1}\right)^{n\beta} \exp\left(-\sum_{i=1}^n x_i \left(\frac{1}{\theta_1} - \frac{1}{\theta_0}\right)\right) > K_\alpha \\
& n\beta(\ln\theta_0 - \ln\theta_1) + \sum_{i=1}^n x_i \underbrace{\left(\frac{1}{\theta_0} - \frac{1}{\theta_1}\right)}_{>0} > \ln K_\alpha \\
& T_n = \sum_{i=1}^n x_i > t_\alpha = \frac{\theta_0\theta_1}{\theta_1 - \theta_0} \left(\ln K_\alpha + n\beta(\ln\theta_1 - \ln\theta_0)\right)
\end{aligned}$$

   (b) It immediately follows that under $H_0$, $T_n \sim \mathcal{IG}(n\beta, \theta_0)$ and under $H_1$, $T_n \sim \mathcal{IG}(n\beta, \theta_1)$.

   (c)

$$\begin{aligned}
\alpha &= P[\text{reject } H_0 | H_0 \text{ true}] = P[T_n > t_\alpha | T_n \sim \mathcal{IG}(n\beta, \theta_0)] \\
&= 1 - \int_0^{t_\alpha} \frac{\theta_0^{n\beta} x^{-n\beta-1}}{\Gamma(n\beta)} \exp(-\theta_0/x) dx = 1 - F_0(t_\alpha) \\
&\Longrightarrow t_\alpha = F_0^{-1}(1-\alpha)
\end{aligned}$$

   (d)

$$\begin{aligned}
\pi &= P[\text{reject } H_0 | H_1 \text{ true}] = P[T_n > t_\alpha | T_n \sim \mathcal{IG}(n\beta, \theta_1)] \\
&= 1 - \int_0^{t_\alpha} \frac{\theta_1^{n\beta} x^{-n\beta-1}}{\Gamma(n\beta)} \exp(-\theta_1/x) dx = 1 - F_1(t_\alpha) \\
&= 1 - F_1(F_0^{-1}(1-\alpha))
\end{aligned}$$

2.

$$\begin{aligned}
\text{reject } H_0 \text{ if} \quad & \frac{\sup_{\theta\in\Omega_1} f(x_1, \cdots, x_n | \theta)}{\sup_{\theta\in\Omega_0} f(x_1, \cdots, x_n | \theta)} = \frac{f(x_1, \cdots, x_n | \theta = \hat{\theta}_{ML(1)})}{f(x_1, \cdots, x_n | \theta = \theta_0)} > K_\alpha \\
& \frac{\prod_{i=1}^n x_i^{\beta-1} \Gamma(\beta)^n \theta_0^{n\beta} \exp\left(-\frac{\sum_{i=1}^n x_i}{\hat{\theta}_{ML(1)}}\right)}{\prod_{i=1}^n x_i^{\beta-1} \Gamma(\beta)^n \hat{\theta}_{ML(1)}^{n\beta} \exp\left(-\frac{\sum_{i=1}^n x_i}{\theta_0}\right)} > K_\alpha \\
& \left(\frac{\beta\theta_0}{\bar{x}}\right)^{n\beta} \exp\left(-n\bar{x}\left(\frac{\beta}{\bar{x}} - \frac{1}{\theta_0}\right)\right) > K_\alpha \\
& n\beta(\ln(\beta\theta_0) - \ln\bar{x}) - n\beta + \frac{n\bar{x}}{\theta_0} > \ln K_\alpha \\
& \beta\theta_0(\ln(\beta\theta_0) - \ln\bar{x} - 1) + \bar{x} > \frac{\theta_0 \ln K_\alpha}{n} \\
& T_n = \bar{x} - \beta\theta_0 \ln\bar{x} > t_\alpha = \frac{\theta_0 \ln K_\alpha}{n} - \beta\theta_0(\ln(\beta\theta_0) - 1)
\end{aligned}$$

3. (a) Since the parameter $\theta$ is unknown, the Kolmogorov test can not be applied. The $\chi^2$ test is appropriate for this problem.

(b)

$$\hat{\theta}_{ML} = \frac{\bar{x}}{\beta} = \frac{\sum_{i=1}^{n} x_i}{n\beta} = \frac{\sum_{i=1}^{25} x_i}{100} = 1.84$$

The mean is given by $\mu = \beta\hat{\theta}_{ML} = 7.36$ and the standard deviation by $\sigma = \sqrt{\beta}\hat{\theta}_{ML} = 3.68$.

(c) With $a = 1 - F^{-1}(3/4)$, the classes for testing for the standard Normal distribution $\mathcal{N}(0,1)$ are given by

$$(-\infty, -a],\ (-a, 0]\ (0, a],\ (a, \infty)$$

and the classes for $\mathcal{N}(\mu = 7.36, \sigma^2 = 3.68^2)$ are therefore

$$\begin{aligned} C_1 &= (-\infty, \mu - \sigma a] = (-\infty, 4.88] \\ C_2 &= (\mu - \sigma a, \mu] = (4.88, 7.36] \\ C_3 &= (\mu, \mu + \sigma a] = (7.36, 9.84] \\ C_4 &= (\mu + \sigma a, \infty] = (9.84, , \infty] \end{aligned}$$

(d) The test statistic is given by

$$\phi = \sum_{k=1}^{K} \frac{(n_k - Np_k)^2}{Np_k}$$

with $K = 4$, $p_k = 1/4$ and $(n_1, n_2, n_3, n_4) = (5, 11, 4, 5)$. Hence,

$$\phi = \frac{4}{25}\left(\frac{25}{16} + \frac{361}{16} + \frac{81}{16} + \frac{25}{16}\right) = \frac{492}{100} = 4.92.$$

Since the unknown parameter $\theta$ was replaced with its maximum likelihood estimate, the critical value of the test is given by the quantile of the $\chi^2$ distribution with $N = K - 1\ - 1$ degrees of freedom, $t_\alpha = 4.61 < 4.92$. The test therefore rejects the hypothesis that the sample is Normal with significance $\alpha = 0.1$.