EID 2015 EXERCISE 1 ESTIMATION

Weibull $f(x) = k\frac{x^{k-1}}{\theta}\exp(-\frac{x^k}{\theta})$. k known.

① ML

a) $LH \propto \theta^{-n} \prod_i x_i^{k-1} \; e^{-\frac{\sum x_i^k}{\theta}}$

$\log \propto -n\log\theta + \frac{1}{\theta}\sum x_i^k$

$\frac{\partial}{\partial\theta} \propto -\frac{n}{\theta} + \frac{1}{\theta^2}\sum x_i^k \quad \Rightarrow \hat{\theta}_{ML} = \frac{1}{n}\sum x_i^k$

$\frac{\partial^2}{\partial\theta^2} = \frac{n}{\theta^2} - \frac{2}{\theta^3}\sum x_i^k$

b) $U = X^k \Rightarrow U \sim G(1,\theta)$

$\rightarrow$ [2013, Ex. 2.1(b)]

$\Rightarrow \mathbb{E}[\hat{\theta}_{ML}] = \frac{1}{n} n\cdot\theta = \theta$ unbiased

$\Rightarrow Var[\hat{\theta}_{ML}] = \frac{1}{n^2} n\theta^2 = \frac{1}{n}\theta^2$

c) CRB

$\mathbb{I} = \mathbb{E}[-\frac{\partial^2 \ell}{\partial\theta^2}] = -\frac{n}{\theta^2} + \frac{2}{\theta^3}\underbrace{\mathbb{E}[\sum x_i^k]}_{n\theta} = \frac{n}{\theta^2} \Rightarrow CRB(\theta) = \frac{\theta^2}{n}$

$\Rightarrow$ efficient

② MoM for θ using $U = X^k$

$m_1 = \mathbb{E}[X^k] = \theta \Rightarrow \hat{\theta}_{m_1} = \frac{1}{n}\sum X^k = \hat{\theta}_{ML}$

$m_2 = \mathbb{E}[X^{2k}] = 2\theta^2 \Rightarrow \hat{\theta}_{m_2} = \sqrt{\frac{1}{2}\sum x_i^{2k}}$

$n=1$: $\hat{\theta}_{m_2} = \frac{x_i^k}{\sqrt{2}} \Rightarrow \mathbb{E}[\hat{\theta}_{m_2}] = \theta/\sqrt{2} \Rightarrow$ bias $\frac{1}{\sqrt{2}} - 1$

$Var[\hat{\theta}_{m_2}] = \theta^2/2$

$mse = 2\theta^2(1 - \frac{1}{\sqrt{2}}) =$

③ Bayes $U = X^k$, $\theta \sim IG(a,b)$

$\rightarrow$ [2012, Ex 1.4]

E/D EXAM 2015 EX 2 DETECTION

$x_1, \dots, x_n \sim \text{Levy}(\mu, c>0)$; μ known

$$p(x) = \sqrt{\frac{\gamma}{2\pi}} \frac{e^{-\frac{\gamma}{2(x-\mu)}}}{(x-\mu)^{\frac{3}{2}}}$$

$\boxed{1}$ $H_0: \gamma = \gamma_0$. $H_1: \gamma = \gamma_1$. $\gamma_1 > \gamma_0$

a) Neyman-Pearson T

LH $p(\{x_i\}) \propto \left(\frac{\gamma}{2\pi}\right)^{\frac{n}{2}} \prod_i (x_i-\mu)^{\frac{3}{2}} e^{-\frac{\gamma}{2}\sum_i (x_i-\mu)^{-1}}$

reject if $\frac{\gamma_1^{\frac{n}{2}} e^{-\frac{\gamma_1}{2}\sum_i (x_i-\mu)^{-1}}}{\gamma_0^{\frac{n}{2}} e^{-\frac{\gamma_0}{2}\sum_i (x_i-\mu)^{-1}}} > t_\alpha$

$$\frac{n}{2}(\ln(\gamma_1) - \ln(\gamma_0)) - \frac{1}{2}(\gamma_1 - \gamma_0) \sum_i (x_i-\mu)^{-1} > t_\alpha$$

$$\underline{T = \sum_i (x_i-\mu)^{-1}} < t_\alpha.$$

b) Law of T

i) Law of $Y = \frac{1}{X-\mu} = g(x) \sim G\left(\frac{1}{2}, \frac{2}{\gamma}\right)$ (note: $\Gamma(\frac{1}{2}) = \sqrt{\pi}$)

$X = g^{-1}(Y) = \frac{1}{Y} + \mu$, $\frac{dY}{dX} = -\frac{1}{Y^2}$

$$f_Y(y) = \left|-\frac{1}{Y^2}\right| \sqrt{\frac{\gamma}{2\pi}} \left(\frac{1}{Y} + \mu - \mu\right)^{-\frac{3}{2}} e^{-\frac{\gamma}{2}\left(\frac{1}{Y} + \mu - \mu\right)} = \frac{1}{\sqrt{\pi}} \left(\frac{2}{\gamma}\right)^{-\frac{1}{2}} \overbrace{y^{-2} y^{\frac{3}{2}}}^{\frac{1}{2}} e^{-\frac{y}{(\gamma/2)}}$$

$$= \frac{1}{\Gamma(\frac{1}{2})(\frac{2}{\gamma})^{\frac{1}{2}}} y^{\frac{1}{2}-1} e^{-\frac{y}{\gamma/2}} \propto \underline{G\left(\frac{1}{2}, \frac{2}{\gamma}\right)}$$

ii) $T \sim G(\frac{n}{2}, \frac{2}{\gamma})$

$T = \sum Y_i$; $\varphi_T(t) = \mathbb{E}[e^{it\sum Y_k}] = \prod_k \mathbb{E}[e^{itY_k}] = (1 - it\theta)^{-nk}$

$$\Rightarrow \boxed{T \propto G\left(\frac{n}{2}, \frac{2}{\gamma}\right)}$$

c) $\alpha = P[\text{reject } H_0 | H_0 \text{ true}] = P\left[T < t_\alpha \,|\, T \sim G\left(\frac{n}{2}, \frac{2}{\gamma_0}\right)\right] = \int_0^{t_\alpha} \frac{1}{\Gamma(\frac{n}{2})(\frac{2}{\gamma_0})^{\frac{n}{2}}} x^{\frac{n}{2}-1} e^{-\frac{x\gamma_0}{2}} dx = F_0(t_\alpha)$

$\Rightarrow t_\alpha = F_0^{-1}(\alpha)$ CDF $G(\frac{n}{2}, \frac{2}{\gamma_0})$

d) $\pi = P[\text{reject } H_0 | H_1 \text{ true}] = \int_0^{t_\alpha} \frac{1}{\Gamma(\frac{n}{2})(\frac{2}{\gamma_1})^{n/2}} x^{\frac{n}{2}-1} e^{-\frac{x\gamma_1}{2}} dx = F_1(t_\alpha) = \underline{F_1(F_0^{-1}(t_\alpha))}$

$\boxed{2}$ $\gamma = \gamma_0$ vs $\gamma \neq \gamma_0$

log LH $\propto \frac{n}{2}\log(\gamma) - \frac{\gamma}{2}\sum_i (x_i-\mu)^{-1}$; $\frac{\partial}{\partial\gamma} = \frac{n}{2\gamma} - \frac{1}{2}\sum (x_i-\mu)^{-1} \Rightarrow \underline{\hat{\gamma} = \frac{n}{\sum (x_i-\mu)^{-1}}}$

reject if $\frac{\left(\frac{n}{\sum(x_i-\mu)^{-1}}\right)^{\frac{n}{2}} e^{-\frac{n}{2\sum(x_i-\mu)^{-1}}\sum(x_i-\mu)^{-1}}}{\gamma_0^{\frac{n}{2}} e^{-\frac{\gamma_0}{2}\sum(x_i-\mu)^{-1}}} > t_\alpha$

$$-\frac{n}{2}\log\left(\sum(x_i-\mu)^{-1}\right) - \frac{\gamma_0}{2}\sum(x_i-\mu)^{-1} > \tilde{t}_\alpha$$

$$\underline{T = \log\left(\sum(x_i-\mu)^{-1}\right) - \frac{\gamma_0}{n}\sum(x_i-\mu)^{-1} < t_\alpha}$$

$\boxed{3}$ $T \sim G(k,\Theta) \approx \mathcal{N}(m,\sigma^2)$ for k large.

here, $k = \frac{n}{2}$

$f(t_1,\dots,t_M) \propto \frac{\prod_{m=1}^{M} t_m^{k-1}}{\Theta^{kM}} e^{-\frac{1}{\Theta}\sum_{m=1}^{M} t_m}$

$\ln \qquad \propto -kM\ln(\Theta) - \frac{1}{\Theta}\sum_{m=1}^{M} t_m$

$\frac{\partial}{\partial\Theta} \propto -\frac{kM}{\Theta} + \frac{1}{\Theta^2}\sum_{m=1}^{M} t_m \overset{!}{=} 0 \Rightarrow \hat{\Theta}_{ML} = \frac{1}{kM}\sum t_m$

suppose $n=32$, $M=25$ $(\Rightarrow k=16)$

$(t_1,\dots,t_M) = (20, 28, 16, 36, 36, 68, 32, 20, 28, 20, 24, 24, 24, 36, 52, 56, 28, 20, 36, 64, 20, 12, 60, 12, 28)$

$\sum t_m = 800$, $\hat{\Theta}_{ML} = \frac{1}{16\cdot 25}\cdot 800 = 2$

$\Rightarrow m = k\Theta = 32$, $\sigma = 4\cdot 2 = 8$

Remainder: corr. 2014 $\times 4$

$C_1 = (-\infty, \mu-\sigma a] = (-\infty, 21.2]$

$C_2 = (\mu-\sigma a, \mu] = (21.2, 32]$

$C_3 = (\mu, \mu+\sigma a] = (32, 42.8]$

$C_4 = (\mu+\sigma a, +\infty) = (42.8, \infty]$

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