EXAM ESTIMATION - DETECTION

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only allowed material: one A4 page of own notes

Exercise 1: Estimation

We consider n independent identically distributed random variables $X_1, \cdots, X_n$ with law given by the exponential distribution $\mathcal{E}(\lambda)$.

1. Maximum likelihood estimation
   (a) Express the likelihood of n observation $(x_1, \cdots, x_n)$ and derive the maximum likelihood estimator $\hat{\lambda}_{ML}$ for λ.
   (b) Using the characteristic function of $\mathcal{E}(\lambda)$, show that if $X_i \sim \mathcal{E}(\lambda)$, then $Y = \sum_{i=1}^{n} X_i \sim \mathcal{G}(n, 1/\lambda)$.
   (c) Use the result given in 1.(b) to determine the bias and variance of $\hat{\lambda}_{ML}$. Is $\hat{\lambda}_{ML}$ unbiased and convergent?
   (d) Determine the Cramer-Rao bound. Is $\hat{\lambda}_{ML}$ efficient?
2. Consider the alternative estimator $\hat{\lambda}_2 = \frac{n-1}{\sum_{i=1}^{n} x_i}$.
   (a) Determine the bias and variance of $\hat{\lambda}_2$. Is $\hat{\lambda}_2$ the efficient estimator of λ?
   (b) Which of the two estimators $\hat{\lambda}_{ML}$ and $\hat{\lambda}_2$ would you prefer and why?
3. Let $\beta = \frac{1}{\lambda} > 0$ and use β instead of λ as the parameter of the exponential distribution.
   (a) Express the likelihood of the n observation $(x_1, \cdots, x_n)$ and derive the maximum likelihood estimator $\hat{\beta}_{ML}$ for β.
   (b) Use the result given in 1.(b) to determine the bias and variance of $\hat{\beta}_{ML}$.
   (c) Is $\hat{\beta}_{ML}$ the efficient estimator for β?
4. Bayesian estimation with Jeffrey's prior for λ
   (a) Derive Jeffrey's (non-informative) prior $p(\theta) \propto \sqrt{I(\lambda)}$ for θ, where $I(\lambda)$ is the Fisher information.
   (b) Determine the posterior law of λ, derive the MMSE and MAP estimators $\hat{\lambda}^J_{MMSE}$ and $\hat{\lambda}^J_{MAP}$ for λ and compare them to $\hat{\lambda}_{ML}$ and $\hat{\lambda}_2$.
5. Bayesian estimation with Gamma prior $\mathcal{G}(k, \theta)$ for λ: $\quad \lambda \sim \mathcal{G}(k, \theta)$
   (a) Derive the posterior law, show that it is $\mathcal{G}(a, b)$ and determine its parameters.
   (b) Determine the MMSE and MAP estimators $\hat{\lambda}_{MMSE}$ and $\hat{\lambda}_{MAP}$ for λ. How do they compare to $\hat{\lambda}_{ML}$ and $\hat{\lambda}_2$ when $n \to \infty$?

Exercise 2: Detection

The lifetime of an electronic component is a random variable that is often modelled by a Weibull distribution $\mathcal{W}(k, \theta)$ with shape parameter $k > 0$ and scale parameter $\theta > 0$. We consider a sample $X_1, \cdots, X_n$, $X_i \overset{i.i.d.}{\sim} \mathcal{W}(k, \theta)$. Suppose that the shape parameter k is known.

1. We want to test the hypotheses $H_0 : \theta = \theta_0$ and $H_1 : \theta = \theta_1$ with $\theta_1 > \theta_0$.
   (a) Determine the test statistic of the Neyman-Pearson test, denoted by T.
   (b) Determine the law of the test statistic T under H_0 and under H_1:
      i. Using the change of variable $U = X^k$, show that U is distributed according to Gamma distribution with parameters $\beta = 1$ and θ, $U \sim \mathcal{G}(1, \theta)$.
      ii. Using the characteristic function of the Gamma distribution, show that $T \sim \mathcal{G}(n, \theta)$.
   (c) Determine the integral equation for the significance α of the test and give an expression for the critical value t_α.
   (d) Determine the power π of the test.
2. We want to test the hypotheses $H_0 : \theta = \theta_0$ and $H_1 : \theta \neq \theta_0$. Determine the test statistic T_{GLR} of the generalized likelihood ratio test.
3. When n is large, the law of the statistic $T \sim \mathcal{G}(n, \theta)$ can be approximated by a Normal law $\mathcal{N}(\mu, \sigma^2)$. We want to test if we can use this approximation, i.e., we want to test the hypotheses

$$H_0 : T \sim \mathcal{N}(\mu, \sigma^2), \qquad H_1 : \text{ not } H_0.$$

   Suppose that $n = 25$ and that we have $M = 25$ observations T_m, $m = 1, \ldots, M$ of the test statistic given by [1]:

   53, 51, 57, 54, 50, 68, 49, 51, 60, 42, 37, 58, 59, 59, 56, 37, 63, 55, 50, 37, 48, 49, 32, 40, 35

   (a) Which test is appropriate for this problem and why?
   (b) Calculate the maximum likelihood estimate for θ and use it to determine μ and σ^2.
   (c) Define the classes for the test with $K = 4$ equi-probable classes (note: $F^{-1}(0.75) = 0.675$, where F is the cumulative distribution function of the standard Normal distribution).
   (d) Perform the test for $\alpha = 0.1$. The quantiles of the chi-square distribution with N degrees of freedom are given by:

N	1	2	3	4	5
$\left(\chi^2_N\right)^{-1}(0.9)$	2.71	4.61	6.25	7.78	9.24

- *Exponential distribution* $\mathcal{E}(\lambda)$ *:* $\lambda > 0$, $x > 0$
  - *density* $f(x) = \lambda \exp(-\lambda x)$
  - *characteristic function* $\varphi_X(t) = \mathbb{E}\left[e^{itX}\right] = (1 - it/\lambda)^{-1}$
- *Weibull distribution* $\mathcal{W}(k, \theta)$ *:* $k > 0$, $\theta > 0$, $x > 0$
  - *density* $f(x) = k\frac{x^{k-1}}{\theta} \exp\left(-\frac{x^k}{\theta}\right)$
- *Gamma distribution* $\mathcal{G}(\beta, \theta)$*:* $\beta > 0$, $\theta > 0$, $x > 0$
  - *density* $f(x; \beta, \theta) = \frac{1}{\Gamma(\beta)\theta^\beta} x^{\beta-1} \exp\left(-\frac{x}{\theta}\right)$
  - *mean* $m = \beta\theta$, *variance* $\nu^2 = \beta\theta^2$
  - *characteristic function* $\varphi_X(t) = \mathbb{E}\left[e^{itX}\right] = (1 - it\theta)^{-\beta}$

[1] *For convenience, the values of the continuous random variables X_i have been rounded to integer values here.*