EXAM ESTIMATION - DETECTION

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only allowed material: one A4 page of own notes

Exercise 1: Estimation

We consider n independent identically distributed random variables $X_1, \cdots, X_n$ from a Gamma law $\mathcal{G}(\beta, \theta)$. Suppose that the shape parameter β is known.

1. Maximum likelihood estimation
   (a) Express the likelihood of n observation $(x_1, \cdots, x_n)$ and derive the maximum likelihood estimator $\hat{\theta}_{ML}$ for θ.
   (b) Determine the bias and variance of $\hat{\theta}_{ML}$. Is $\hat{\theta}_{ML}$ unbiased and convergent?
   (c) Determine the Cramer-Rao bound for an unbiased estimator of θ. Is $\hat{\theta}_{ML}$ the efficient estimator for θ?
   (d) Determine which law $\hat{\theta}_{ML}$ follows:
      - Using the characteristic function, show that if $X_i \sim \mathcal{G}(a, b)$ are i.i.d. random variables, then $Y = \sum_{i=1}^{n} X_i \sim \mathcal{G}(na, b)$.
      - Show that if $X \sim \mathcal{G}(a, b)$, then $Z = cX \sim \mathcal{G}(a, cb)$.

2. Method of moments
   (a) Derive an estimator for θ using the first moment of X, denoted by $\hat{\theta}_{m_1}$, determine if it is biased, if it is convergent and if it is the efficient estimator for θ.
   (b) Derive an estimator for θ using the second moment of X, denoted by $\hat{\theta}_{m_2}$.
   (c) Now let $n = 1$ and determine the bias, variance and mean squared error of $\hat{\theta}_{m_2}$.

3. Bayesian estimation with Jeffrey's prior for θ
   (a) Derive Jeffrey's (non-informative) prior $p(\theta) \propto \sqrt{I(\theta)}$ for θ, where $I(\theta)$ is the Fisher information.
   (b) Determine the posterior law of θ and derive the MAP estimator $\hat{\theta}^J_{MAP}$ for θ. Express $\hat{\theta}^J_{MAP}$ in terms of $\hat{\theta}_{ML}$.

4. Bayesian estimation with inverse Gamma prior $\mathcal{IG}(k, \tau)$ for θ: $\quad \theta \sim \mathcal{IG}(k, \tau)$
   (a) Derive the posterior law, show that it is $\mathcal{IG}(a, b)$ and determine its parameters.
   (b) Derive the MAP estimator $\hat{\theta}_{MAP}$ for θ.
   (c) Show that the expectation of an inverse Gamma random variable $X \sim \mathcal{IG}(k, \tau)$ is given by $\mathbb{E}[X] = \frac{\tau}{k-1}$ and determine the MMSE estimator $\hat{\theta}_{MMSE}$ for θ.

Exercise 2: Detection

As in the previous exercise, we consider n independent identically distributed random variables $X_1, \cdots, X_n$ from a Gamma law $\mathcal{G}(\beta, \theta)$ with known shape parameter β.

1. We want to test the hypotheses $H_0 : \theta = \theta_0$ and $H_1 : \theta = \theta_1$ with $\theta_1 > \theta_0$.
   (a) Determine the test statistic of the Neyman-Pearson test, denoted by T_n.
   (b) Using the results from Exercise 1, 1.(d), determine the law of T_n under H_0 and under H_1.
   (c) Determine the integral equation for the significance α of the test and give an expression for the critical value t_α.
   (d) Determine the power π of the test.
2. We want to test the hypotheses $H_0 : \theta = \theta_0$ and $H_1 : \theta \neq \theta_0$. Determine the test statistic of the generalized likelihood ratio test.
3. The random variables $X_i \sim \mathcal{G}(\beta, \theta)$ can be expressed as $X_i = \sum_{l=1}^{L} Y_l$ with i.i.d. random variables $Y_l \sim \mathcal{G}(\beta/L, \theta)$. Therefore, the central limit theorem applies. Motivated by this fact, we want to test whether the law of X_i can be approximated by a Normal law,

$$H_0 : X_i \sim \mathcal{N}(\mu, \sigma^2), \qquad H_1 : \text{ not } H_0.$$

   Suppose that $\beta = 4$ and that we have $n = 25$ observations $x_1, \cdots, x_n$ given by[1]:

$$5,\ 7,\ 4,\ 7,\ 9,\ 13,\ 8,\ 4,\ 6,\ 5,\ 6,\ 6,\ 6,\ 9,\ 11,\ 14,\ 7,\ 3,\ 8,\ 15,\ 5,\ 3,\ 15,\ 3,\ 5$$

   (a) Which test is appropriate for this problem and why?
   (b) Calculate the maximum likelihood estimate of θ for X_i and use it to determine $\mu = \beta\hat{\theta}_{ML}$ and $\sigma = \sqrt{\beta\hat{\theta}_{ML}^2}$.
   (c) Define the classes for the test with $K = 4$ equi-probable classes (note: $F^{-1}(0.75) = 0.675$, where F is the cdf of the standard Normal distribution).
   (d) Perform the test for $\alpha = 0.1$. The quantiles of the chi-square distribution with N degrees of freedom are given by:

N	1	2	3	4	5
$\left(\chi^2_N\right)^{-1}(0.9)$	2.71	4.61	6.25	7.78	9.24

- *Gamma distribution* $\mathcal{G}(\beta, \theta)$: $\quad \beta > 0, \theta > 0, x > 0$
  - *density* $f(x; \beta, \theta) = \frac{1}{\Gamma(\beta)\theta^\beta} x^{\beta-1} \exp\left(-\frac{x}{\theta}\right)$
  - *mean* $m = \beta\theta$
  - *variance* $\nu^2 = \beta\theta^2$
  - *characteristic function* $\varphi_X(t) = \mathbb{E}\left[e^{itX}\right] = (1 - it\theta)^{-\beta}$
- *Inverse Gamma distribution* $\mathcal{IG}(a, b)$ *:* $\quad a > 0, b > 0, x > 0$
  - *density* $f(x; a, b) = \frac{b^a}{\Gamma(a)} x^{-a-1} \exp\left(-\frac{b}{x}\right)$

[1] *For convenience, the values of the continuous random variables X_i have been rounded to integer values here.*