ESTIMATION - DETECTION

TD 2 —Detection — SOLUTIONS

Exercise 1

1.(a)

$$\text{reject } H_0 \text{ if} \quad \frac{f(x_1,\cdots,x_n|H_1)}{f(x_1,\cdots,x_n|H_0)} > K_\alpha$$

$$\frac{\frac{1}{(2\pi\sigma^2)^{n/2}}\exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^n (x_i-\tau)^2\right)}{\frac{1}{(2\pi\sigma^2)^{n/2}}\exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^n x_i^2\right)} > K_\alpha$$

$$\sum_{i=1}^n x_i^2 - \sum_{i=1}^n (x_i-\tau)^2 > 2\sigma^2 \ln K_\alpha$$

$$T = \sum_{i=1}^n x_i > t_\alpha$$

The critical region of the test is given by $\Delta = \{(x_1,\cdots,x_n)|\sum_{i=1}^n x_i > t_\alpha\}$. *(Note that the sample mean could be used equivalently as the test statistic.)*

1.(b)

$$\begin{aligned}
\alpha &= P[\text{reject } H_0|H_0 \text{ true}] = P\left[T > t_\alpha | x_i \sim \mathcal{N}\left(0,\sigma^2\right)\right] = P\left[T > t_\alpha | T \sim \mathcal{N}\left(0, n\sigma^2\right)\right] \\
&= P\left[U = \frac{T}{\sqrt{n}\sigma} > \frac{t_\alpha}{\sqrt{n}\sigma} | U \sim \mathcal{N}(0,1)\right] = \int_{\frac{t_\alpha}{\sqrt{n}\sigma}}^\infty \frac{1}{\sqrt{2\pi}}\exp(-u^2/2)du \\
&= 1 - F\left(\frac{t_\alpha}{\sqrt{n}\sigma}\right) = \phi\left(\frac{t_\alpha}{\sqrt{n}\sigma}\right) \quad \Longrightarrow \quad t_\alpha = F^{-1}(1-\alpha)\sqrt{n}\sigma = \phi^{-1}(\alpha)\sqrt{n}\sigma \\
\pi &= P[\text{reject } H_0|H_1 \text{ true}] = P\left[T > t_\alpha | T \sim \mathcal{N}\left(n\tau, n\sigma^2\right)\right] \\
&= P\left[W = \frac{T-n\tau}{\sqrt{n}\sigma} > \frac{t_\alpha - n\tau}{\sqrt{n}\sigma} | V \sim \mathcal{N}(0,1)\right] = \int_{\frac{t_\alpha - n\tau}{\sqrt{n}\sigma}}^\infty \frac{1}{\sqrt{2\pi}}\exp(-u^2/2)du \\
&= \phi\left(\frac{t_\alpha - n\tau}{\sqrt{n}\sigma}\right) = \phi\left(\phi^{-1}(\alpha) - \frac{\sqrt{n}\tau}{\sigma}\right)
\end{aligned}$$

1.(c) $\pi = \frac{1}{2} \implies \phi^{-1}(\alpha) - \frac{\sqrt{n}\tau}{\sigma} = 0 \implies \tau = \frac{\sigma}{\sqrt{n}}\phi^{-1}(\alpha)$

2.(a) The hypotheses are $H_0 : \tau \in \Omega_0 = \{0\}$ and $H_1 : \tau \in \Omega_1 = \mathbb{R}^*$. The maximum likelihood estimator for τ under H_1 is given by the sample mean $\hat{\tau}_1^{ML} = \frac{1}{n}\sum_{i=1}^n x_i = \bar{x}$. The generalized likelihood ratio test reads

$$\text{reject } H_0 \text{ if} \quad \frac{\sup_{\tau\in\Omega_1} f(x_1,\cdots,x_n|\tau)}{\sup_{\tau\in\Omega_0} f(x_1,\cdots,x_n|\tau)} = \frac{f(x_1,\cdots,x_n|\tau = \hat{\tau}_1^{ML})}{f(x_1,\cdots,x_n|\tau=0)} > K_\alpha$$

$$\frac{\frac{1}{(2\pi\sigma^2)^{n/2}}\exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^n (x_i-\bar{x})^2\right)}{\frac{1}{(2\pi\sigma^2)^{n/2}}\exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^n x_i^2\right)} > K_\alpha$$

$$\sum_{i=1}^n x_i^2 - \sum_{i=1}^n (x_i-\bar{x})^2 > 2\sigma^2 \ln K_\alpha$$

$$\bar{x}^2 > \frac{2\sigma^2 \ln K_\alpha}{n}$$

$$T = |\bar{x}| > \sqrt{\frac{2\sigma^2 \ln K_\alpha}{n}} = t_\alpha$$

$$\begin{aligned}
\alpha &= P[\text{reject } H_0|H_0 \text{ true}] = P\left[|\bar{x}| > t_\alpha | \bar{x} \sim \mathcal{N}\left(0, \frac{\sigma^2}{n}\right)\right] = 2P\left[\bar{x} > t_\alpha | \bar{x} \sim \mathcal{N}\left(0, \frac{\sigma^2}{n}\right)\right] \\
&= 2P\left[W = \frac{\bar{x}}{\sigma/\sqrt{n}} > \frac{t_\alpha}{\sigma/\sqrt{n}} | W \sim \mathcal{N}(0,1)\right] = 2\int_{\frac{t_\alpha}{\sigma/\sqrt{n}}}^\infty \frac{\exp(-u^2/2)}{\sqrt{2\pi}}du = 2\phi\left(\frac{t_\alpha\sqrt{n}}{\sigma}\right) \\
&\Longrightarrow \quad t_\alpha = \phi^{-1}\left(\frac{\alpha}{2}\right)\frac{\sigma}{\sqrt{n}}
\end{aligned}$$

2.(b)

$$\pi(\tau^*) = P\left[|\bar{x}| > t_\alpha|\bar{x} \sim \mathcal{N}\left(\tau^*, \frac{\sigma^2}{n}\right)\right] = P\left[\bar{x} > t_\alpha|\bar{x} \sim \mathcal{N}\left(\tau^*, \frac{\sigma^2}{n}\right)\right] + P\left[\bar{x} < -t_\alpha|\bar{x} \sim \mathcal{N}\left(\tau^*, \frac{\sigma^2}{n}\right)\right]$$

$$= P\left[W = \frac{\bar{x}-\tau^*}{\sigma/\sqrt{n}} > \frac{t_\alpha - \tau^*}{\sigma/\sqrt{n}}|W \sim \mathcal{N}(0,1)\right] + P\left[W = \frac{\bar{x}-\tau^*}{\sigma/\sqrt{n}} < \frac{-t_\alpha - \tau^*}{\sigma/\sqrt{n}}|W \sim \mathcal{N}(0,1)\right]$$

$$= 1 + \phi\left(\frac{(t_\alpha - \tau^*)\sqrt{n}}{\sigma}\right) - \phi\left(\frac{-(t_\alpha + \tau^*)\sqrt{n}}{\sigma}\right) = 1 + \phi\left(\phi^{-1}\left(\frac{\alpha}{2}\right) - \frac{\tau^*\sqrt{n}}{\sigma}\right) - \phi\left(-\phi^{-1}\left(\frac{\alpha}{2}\right) - \frac{\tau^*\sqrt{n}}{\sigma}\right)$$

Note that $\pi(\tau^*)$ can be interpreted here as the probability of detection of a target of "size" τ^* (i.e., a target giving rise to reflections of power proportional to τ^{*2} in the observations x_i).

3. The Bayesian test is given by

$$\text{reject } H_0 \text{ if} \quad \frac{f(x_1,\cdots,x_n|H_1)}{f(x_1,\cdots,x_n|H_0)} = \frac{\int f(x_1,\cdots,x_n|\tau)p_1(\tau)d\tau}{\int f(x_1,\cdots,x_n|\tau)p_0(\tau)d\tau} > K_\alpha$$

where

$$f(x_1,\cdots,x_n|H_0) = \frac{1}{(2\pi\sigma^2)^{n/2}}\exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^n x_i^2\right)$$

$$f(x_1,\cdots,x_n|H_1) = \int \frac{1}{(2\pi\sigma^2)^{n/2}}\exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^n (x_i-\mu)^2\right)\frac{1}{\sqrt{2\pi\nu^2}}\exp\left(-\frac{\mu^2}{2\nu^2}\right)d\mu$$

$$= \frac{1}{\sqrt{2\pi\nu^2(2\pi\sigma^2)^n}}\int \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^n (x_i-\mu)^2 - \frac{\mu^2}{2\nu^2}\right)d\mu$$

$$= \underbrace{\frac{\exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^n x_i^2\right)}{\sqrt{2\pi\nu^2(2\pi\sigma^2)^n}}}_{c_1}\int \exp\left(-\frac{1}{2\sigma^2}\left(n\mu^2 - 2\mu\sum_{i=1}^n x_i\right) - \frac{\mu^2}{2\nu^2}\right)d\mu$$

$$= c_1\int \exp\left(-\frac{1}{2}\frac{n\nu^2+\sigma^2}{\nu^2\sigma^2}\left(\mu^2 - 2\mu\frac{\bar{x}n\nu^2}{n\nu^2+\sigma^2}\right)\right)d\mu$$

$$= c_1\exp\left(\frac{n\nu^2+\sigma^2}{2\nu^2\sigma^2}\bar{x}^2\left(\frac{n\nu^2}{n\nu^2+\sigma^2}\right)^2\right)\underbrace{\underbrace{\int \exp\left(-\frac{1}{2}\overbrace{\frac{n\nu^2+\sigma^2}{\nu^2\sigma^2}}^{1/\tilde{\sigma}^2}\left(\mu - \overbrace{\frac{\bar{x}n\nu^2}{n\nu^2+\sigma^2}}^{\tilde{m}}\right)^2\right)}_{\propto \mathcal{N}(\tilde{m},\tilde{\sigma}^2)}d\mu}_{=\sqrt{2\pi\tilde{\sigma}^2}=\sqrt{\frac{2\pi\nu^2\sigma^2}{n\nu^2+\sigma^2}}}$$

$$= c_2\exp\left(\bar{x}^2\frac{n^2\nu^2}{2\sigma^2(n\nu^2+\sigma^2)}\right) \qquad \text{where} \quad c_2 = \sqrt{\frac{\sigma^2}{n\nu^2+\sigma^2}}\frac{\exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^n x_i^2\right)}{(2\pi\sigma^2)^{n/2}}$$

Therefore,

$$\text{reject } H_0 \text{ if} \quad \frac{f(x_1,\cdots,x_n|H_1)}{f(x_1,\cdots,x_n|H_0)} = \frac{\int f(x_1,\cdots,x_n|\tau)p_1(\tau)d\tau}{\int f(x_1,\cdots,x_n|\tau)p_0(\tau)d\tau} > K_\alpha$$

$$\sqrt{\frac{\sigma^2}{n\nu^2+\sigma^2}}\exp\left(\bar{x}^2\frac{n^2\nu^2}{2\sigma^2(n\nu^2+\sigma^2)}\right) > K_\alpha$$

$$\bar{x}^2\frac{n^2\nu^2}{2\sigma^2(n\nu^2+\sigma^2)} > \ln\left(K_\alpha\sqrt{\frac{n\nu^2+\sigma^2}{\sigma^2}}\right)$$

$$|\bar{x}| > \sqrt{\ln\left(K_\alpha\sqrt{\frac{n\nu^2+\sigma^2}{\sigma^2}}\right)\frac{2\sigma^2(n\nu^2+\sigma^2)}{n^2\nu^2}}$$

$$T = |\bar{x}| > t_\alpha.$$

The prior $p_1(\tau)$ is non informative, and the test statistic is identical to the test statistic of the GLR test. Note that by choosing a different prior, we would obtain a different test statistic (for instance by choosing a Gaussian prior with non-zero mean).

Exercise 2

1.(a) The likelihood is given by

$$f(x_1,\cdots,x_n;\sigma^2)=\prod_{i=1}^{n}\frac{x_i}{\sigma^2}\exp\left(-\frac{x_i^2}{2\sigma^2}\right)=\frac{1}{(\sigma^2)^n}\prod_{i=1}^{n}x_i\exp\left(-\frac{x_i^2}{2\sigma^2}\right)$$

$$\begin{aligned}\text{reject } H_0 \text{ if} \quad \frac{f(x_1,\cdots,x_n|H_1)}{f(x_1,\cdots,x_n|H_0)}&>K_\alpha\\ \frac{(\sigma_0^2)^n\prod_{i=1}^n x_i\exp\left(-\frac{x_i^2}{2\sigma_1^2}\right)}{(\sigma_1^2)^n\prod_{i=1}^n x_i\exp\left(-\frac{x_i^2}{2\sigma_0^2}\right)}&>K_\alpha\\ \left(\frac{\sigma_0^2}{\sigma_1^2}\right)^n\exp\left(-\sum_{i=1}^n x_i^2\left(\frac{1}{2\sigma_1^2}-\frac{1}{2\sigma_0^2}\right)\right)&>K_\alpha\\ T=\sum_{i=1}^n x_i^2&>\frac{2\sigma_0^2\sigma_1^2}{\sigma_1^2-\sigma_0^2}\ln\left(\left(\frac{\sigma_1^2}{\sigma_0^2}\right)^n K_\alpha\right)=t_\alpha\end{aligned}$$

with critical region $\Delta=\{(x_1,\cdots,x_n)|\sum_{i=1}^n x_i^2>t_\alpha\}$.

1.(b)

$$\begin{aligned}\alpha&=P[\text{reject } H_0|H_0\text{ true}]=P\left[T>t_\alpha|\sigma=\sigma_0\right]=P\left[T>t_\alpha|T\sim\mathcal{G}(n,2\sigma_0^2)\right]\\&=\int_{t_\alpha}^{\infty}\frac{\tau^{n-1}}{\Gamma(n)\sigma_0^{2n}}\exp\left(-\frac{\tau}{2\sigma_0^2}\right)d\tau\\ \pi&=P[\text{reject } H_0|H_1\text{ true}]=P\left[T>t_\alpha|T\sim\mathcal{G}(n,2\sigma_1^2)\right]=\int_{t_\alpha}^{\infty}\frac{\tau^{n-1}}{\Gamma(n)\sigma_1^{2n}}\exp\left(-\frac{\tau}{2\sigma_1^2}\right)d\tau\end{aligned}$$

2.(a)

$$\begin{aligned}\text{reject } H_0\text{ if}\quad\frac{f(x_1,\cdots,x_n|H_1)}{f(x_1,\cdots,x_n|H_0)}&>\frac{P(H_0)}{P(H_1)}\frac{(c_{10}-c_{00})}{(c_{01}-c_{11})}\\ \left(\frac{\sigma_0^2}{\sigma_1^2}\right)^n\exp\left(-\sum_{i=1}^n x_i^2\left(\frac{1}{2\sigma_1^2}-\frac{1}{2\sigma_0^2}\right)\right)&>\frac{P(H_0)}{P(H_1)}\frac{(c_{10}-c_{00})}{(c_{01}-c_{11})}\\ T=\sum_{i=1}^n x_i^2&>\frac{2\sigma_0^2\sigma_1^2}{\sigma_1^2-\sigma_0^2}\ln\left(\left(\frac{\sigma_1^2}{\sigma_0^2}\right)^n\frac{P(H_0)}{P(H_1)}\frac{(c_{10}-c_{00})}{(c_{01}-c_{11})}\right)=t_{crit}\end{aligned}$$

2.(b) The costs are (proportional to) $c_{00}=2$, $c_{01}=4$, $c_{11}=3$, $c_{10}=6$. We find

$$t_{crit}=\frac{2\sigma_0^2\sigma_1^2}{\sigma_1^2-\sigma_0^2}\ln\left(\left(\frac{\sigma_1^2}{\sigma_0^2}\right)^n\frac{P(H_0)}{P(H_1)}\frac{(c_{10}-c_{00})}{(c_{01}-c_{11})}\right)=23.5,$$

therefore $\int_{t_{crit}}^{\infty}f(x|H_0)dx=0.988$ and $\int_{t_{crit}}^{\infty}f(x|H_1)dx=0.632$ and

$$\mathbb{E}[C]=c_{00}P(H_0)+c_{01}P(H_1)+(c_{10}-c_{00})P(H_0)\int_{t_{crit}}^{\infty}f(x|H_0)dx+(c_{11}-c_{10})P(H_1)\int_{t_{crit}}^{\infty}f(x|H_1)dx=2.44.$$

The expected execution time is therefore $2.44s$.

2.(c) We find $t_\alpha=22.7$, $\int_{t_\alpha}^{\infty}f(x|H_0)dx=0.955=1-\alpha$ and $\int_{t_\alpha}^{\infty}f(x|H_1)dx=0.414$ and therefore

$$\mathbb{E}[C]=c_{00}P(H_0)+c_{01}P(H_1)+(c_{10}-c_{00})P(H_0)\int_{t_\alpha}^{\infty}f(x|H_0)dx+(c_{11}-c_{10})P(H_1)\int_{t_\alpha}^{\infty}f(x|H_1)dx=2.53,$$

the expected execution time is $2.53ss$.

3.(a) The mean and variance of T are given by $\mu_{\mathcal{N}} = 2n\sigma^2$ and $\sigma^2_{\mathcal{N}} = 4n\sigma^4$ with $\sigma^2 = 0.01$ and $n = 4096$. With the upper quartiles of the standard Normal distribution given by $a = 0.6745$, this gives the $K = 4$ equi-probable classes

$$\begin{aligned} C_1 &= (-\infty,\, \mu_{\mathcal{N}} - \sigma_{\mathcal{N}} a] = (-\infty,\, 81.1] \\ C_2 &= (\mu_{\mathcal{N}} - \sigma_{\mathcal{N}} a,\, \mu_{\mathcal{N}}] = (81.1,\, 81.9] \\ C_3 &= (\mu_{\mathcal{N}}, \mu_{\mathcal{N}} + \sigma_{\mathcal{N}} a] = (81.9,\, 82.8] \\ C_4 &= (\mu_{\mathcal{N}} + \sigma_{\mathcal{N}} a,\, +\infty) = (82.8,\, \infty) \end{aligned}$$

The χ^2 test statistic is given by $\phi = \sum_{k=1}^{K} \left(\frac{(n_k - Np_k)^2}{Np_k} \right)$, where n_k are the number of observations per class, and p_k are the probabilities of the theoretical law for each class. Here, $Np_k = 20 \cdot 0.25 = 5$, $(n_1, n_2, n_3, n_4) = (4, 5, 5, 6)$ which yields $\phi = 0.4$. Under H_0, ϕ follows a χ^2_{K-1} law. The critical value is hence $t_{\alpha=0.1} = 6.25$, and the test does not reject H_0 with significance $\alpha = 0.1$.

3.(b) We know that $T_n \sim \mathcal{G}(\beta = n, \theta = 2\sigma^2)$ and need to calculate the maximum likelihood estimate for θ:

$$f(x_1, \cdots, x_N; \beta, \theta) = \frac{\prod_{i=1}^{N} x_i^{\beta-1}}{\Gamma(\beta)^N \theta^{N\beta}} \exp\left(-\frac{\sum_{i=1}^{N} x_i}{\theta} \right)$$

$$\ln f(x_1, \cdots, x_N; \beta, \theta) = -N\beta \ln\theta - N \ln\Gamma(\beta) + (\beta - 1) \sum_{i=1}^{N} \ln x_i - \frac{\sum_{i=1}^{N} x_i}{\theta}$$

$$\frac{\partial \ln f(x_1, \cdots, x_N; \beta, \theta)}{\partial\theta} = -\frac{N\beta}{\theta} + \frac{\sum_{i=1}^{N} x_i}{\theta^2} \stackrel{!}{=} 0 \qquad \Longrightarrow \qquad \hat{\theta} = \frac{\sum_{i=1}^{N} x_i}{N\beta}$$

$$\left.\frac{\partial^2 \ln f(x_1, \cdots, x_N; \beta, \theta)}{\partial\theta^2}\right|_{\theta=\bar{x}/\beta} = \left.\frac{N\beta}{\theta^2} - 2\frac{\sum_{i=1}^{N} x_i}{\theta^3}\right|_{\theta=\bar{x}/\beta} = \frac{N\beta}{\bar{x}^2/\beta^2} - 2\frac{N\bar{x}}{\bar{x}^3/\beta^3} = -\frac{N\beta^3}{\bar{x}^2} < 0.$$

Application to the observed test statistics gives

$$\hat{\theta} = \frac{\sum_{i=1}^{N} x_i}{nN} = \frac{1640.1}{4096 \cdot 20}$$

and hence $\mu_{\mathcal{N}} = n\hat{\theta} = 82.0050$ and $\sigma_{\mathcal{N}} = \sqrt{n\hat{\theta}^2} = 1.2813$. This gives the $K = 4$ equi-probable classes

$$\begin{aligned} C_1 &= (-\infty,\, \mu_{\mathcal{N}} - \sigma_{\mathcal{N}} a] = (-\infty,\, 81.1] \\ C_2 &= (\mu_{\mathcal{N}} - \sigma_{\mathcal{N}} a,\, \mu_{\mathcal{N}}] = (81.1,\, 82.0] \\ C_3 &= (\mu_{\mathcal{N}}, \mu_{\mathcal{N}} + \sigma_{\mathcal{N}} a] = (82.0,\, 82.9] \\ C_4 &= (\mu_{\mathcal{N}} + \sigma_{\mathcal{N}} a,\, +\infty) = (82.9,\, \infty) \end{aligned}$$

and $(n_1, n_2, n_3, n_4) = (4, 6, 4, 6)$, which yields $\phi = 0.8$. Under H_0, ϕ follows a χ^2_{K-2} law. The critical value is hence $t_{\alpha=0.1} = 4.61$, and the test does not reject H_0 with significance $\alpha = 0.1$.

3.(c)

$$\text{reject } H_0 \text{ if} \quad \sup_{x\in\mathbb{R}} \left| F_0(x) - \hat{F}(x) \right| = \sup_{i\in\{1,2,\ldots,20\}} \max\left(E_i^-, E_i^+\right) > t_\alpha$$

$$\text{where } E_i^- = \left| F_0(x_i^*) - \frac{i-1}{N} \right| \text{ and } E_i^+ = \left| F_0(x_i^*) - \frac{i}{N} \right|$$

i	1	2	3	4	5	6	7	8	9	10
x_i	79.8	80	80.2	80.9	81.2	81.2	81.5	81.6	81.6	82
F_0	0.05	0.07	0.09	0.21	0.29	0.29	0.37	0.4	0.4	0.52
E_i^+	0	0.03	0.06	0.01	0.04	0.01	0.02	0	0.05	0.02
E_i^-	0.05	0.02	0.01	0.06	0.09	0.04	0.07	0.05	0	0.07
$\max\left(E_i^-, E_i^+\right)$	0.05	0.03	0.06	0.06	0.09	0.04	0.07	0.05	0.05	0.07

i	11	12	13	14	15	16	17	18	19	20
x_i	82.2	82.2	82.2	82.3	83.1	83.1	83.2	83.7	83.8	84.3
F_0	0.59	0.59	0.59	0.62	0.82	0.82	0.84	0.92	0.93	0.97
E_i^+	0.04	0.01	0.06	0.08	0.07	0.02	0.01	0.02	0.02	0.03
E_i^-	0.09	0.04	0.01	0.03	0.12	0.07	0.04	0.07	0.03	0.02
$\max\left(E_i^-, E_i^+\right)$	0.09	0.04	0.06	0.08	0.12	0.07	0.04	0.07	0.03	0.03

Hence, $\sup_{x\in\mathbb{R}} \left| F_0(x) - \hat{F}(x) \right| = 0.12$, and H_0 is not rejected with $\alpha = 0.1$.
NOTE: all values rounded for convenience of pen-and-paper computation...

Exercise 3

1.(a)

$$\text{reject } H_0 \text{ if} \quad \frac{f(x_1, \cdots, x_n|H_1)}{f(x_1, \cdots, x_n|H_0)} > K_\alpha$$

$$\frac{(2\pi\sigma_0^2)^{n/2} \exp\left(\frac{1}{2\sigma_1^2}\sum_{i=1}^n (x_i-\mu)^2\right)}{(2\pi\sigma_1^2)^{n/2} \exp\left(\frac{1}{2\sigma_0^2}\sum_{i=1}^n (x_i-\mu)^2\right)} > K_\alpha$$

$$\left(1 - \frac{\sigma_0^2}{\sigma_1^2}\right)\sum_{i=1}^n \frac{(x_i-\mu)^2}{\sigma_0^2} > \ln \tilde{K}_\alpha$$

$$T_n = \sum_{i=1}^n \frac{(x_i-\mu)^2}{\sigma_0^2} > t_\alpha.$$

We know that if $(Z_1, \cdots, Z_K)$ is a sample with standard Normal law, $Z_k \sim \mathcal{N}(0,1)$, then $\sum_{k=1}^K Z_k^2 \sim \chi_K^2$ and therefore $T_n \sim \chi_n^2$ under H_0. The critical region of the test is $\Delta = \left\{(x_1, \cdots, x_n)|\sum_{i=1}^n \frac{(x_i-\mu)^2}{\sigma_0^2} > t_\alpha\right\}$.

1.(b) $\alpha = P[\text{reject } H_0|H_0 \text{ true}] = P\left[T_n > t_\alpha|\sigma^2 = \sigma_0^2\right] = P\left[T_n > t_\alpha|T_n \sim \chi_n^2\right] = 1 - F_n(t_\alpha)$,

where $F_n(x)$ is the cdf of the χ^2 distribution with n degrees of freedom. The critical value of the test is given by the expression

$$t_\alpha = F_n^{-1}(1-\alpha).$$

Similarly,

$$\pi = P[\text{reject } H_0|H_1 \text{ true}] = P\left[T_n > t_\alpha|\sigma^2 = \sigma_1^2\right]$$
$$= P\left[U = \frac{T_n\sigma_0^2}{\sigma_1^2} > \frac{t_\alpha\sigma_0^2}{\sigma_1^2}|U \sim \chi_n^2\right] = 1 - F_n\left(\frac{\sigma_0^2}{\sigma_1^2}F_n^{-1}(1-\alpha)\right),$$

and $\pi \uparrow$ when $n \uparrow$, $\sigma_1^2 \uparrow$ and $\sigma_0^2 \downarrow$.

2. The ML estimator for the mean is given by the sample mean $\bar{x} = \frac{1}{n}\sum_{i=1}^n x_i$. The GRL test reads

$$\text{reject } H_0 \text{ if} \quad \frac{f(x_1, \cdots, x_n|\sigma = \sigma_1, \mu = \bar{x})}{f(x_1, \cdots, x_n|\sigma = \sigma_0, \mu = \bar{x})} > K_\alpha$$

$$\frac{(2\pi\sigma_0^2)^{n/2} \exp\left(\frac{1}{2\sigma_1^2}\sum_{i=1}^n (x_i-\bar{x})^2\right)}{(2\pi\sigma_1^2)^{n/2} \exp\left(\frac{1}{2\sigma_0^2}\sum_{i=1}^n (x_i-\bar{x})^2\right)} > K_\alpha$$

$$\left(1 - \frac{\sigma_0^2}{\sigma_1^2}\right)\sum_{i=1}^n \frac{(x_i-\bar{x})^2}{\sigma_0^2} > \ln \tilde{K}_\alpha$$

$$T_n = \sum_{i=1}^n \frac{(x_i-\bar{x})^2}{\sigma_0^2} > t_\alpha.$$

Note that $T_n = \sum_{i=1}^n \frac{(x_i-\bar{x})^2}{\sigma_0^2} = \frac{(n-1)s^2}{\sigma_0^2}$, where s^2 is the sample variance, and $T_n \sim \chi_{n-1}^2$. The expressions for α, t_α and π follow from 1.(b).

3.(a)

$$\tilde{\sigma}_{ML}^2 = \frac{1}{n}\sum_{i=1}^n (x_i - m_0)^2$$

3.(b)

$$\hat{m}_{ML} = \bar{x} = \frac{1}{n}\sum_{i=1}^n x_i, \qquad \hat{\sigma}_{ML}^2 = \frac{1}{n}\sum_{i=1}^n (x_i - \bar{x})^2$$

3.(c)

$$\text{reject } H_0 \text{ if} \quad \frac{(2\pi\tilde{\sigma}^2_{ML})^{n/2}\exp\left(-\frac{1}{2\tilde{\sigma}^2_{ML}}\sum_{i=1}^n(x_i-\bar{x})^2\right)}{(2\pi\hat{\sigma}^2_{ML})^{n/2}\exp\left(-\frac{1}{2\hat{\sigma}^2_{ML}}\sum_{i=1}^n(x_i-m_0)^2\right)} > K_\alpha$$

$$\left(\frac{\tilde{\sigma}^2_{ML}}{\hat{\sigma}^2_{ML}}\right)^{n/2}\exp\left(-\frac{1}{2}\left(\frac{\sum_{i=1}^n(x_i-\bar{x})^2}{\hat{\sigma}^2_{ML}}-\frac{\sum_{i=1}^n(x_i-m_0)^2}{\tilde{\sigma}^2_{ML}}\right)\right) > K_\alpha$$

$$\left(\frac{\tilde{\sigma}^2_{ML}}{\hat{\sigma}^2_{ML}}\right)^{n/2}\exp\left(-\frac{1}{2}\left(\frac{n\hat{\sigma}^2_{ML}}{\hat{\sigma}^2_{ML}}-\frac{n\tilde{\sigma}^2_{ML}}{\tilde{\sigma}^2_{ML}}\right)\right) > K_\alpha$$

$$\frac{\tilde{\sigma}^2_{ML}}{\hat{\sigma}^2_{ML}} > K_\alpha$$

$$t_n = \frac{\sum_{i=1}^n(x_i-m_0)^2}{\sum_{i=1}^n(x_i-\bar{x})^2} > t_\alpha$$

3.(d)

$$\sum_{i=1}^n(X_i-\mu)^2 = \sum_{i=1}^n(X_i-\bar{X}+\bar{X}-\mu)^2 = \sum_{i=1}^n(X_i-\bar{X})^2 + \underbrace{\sum_{i=1}^n(\bar{X}-\mu)^2}_{n(\bar{X}-\mu)^2} + 2(\bar{X}-\mu)\underbrace{\sum_{i=1}^n(X_i-\bar{X})}_{(\sum X_i)-n\bar{X}=0}$$

and

$$t_n = \frac{\sum_{i=1}^n(X_i-\mu)^2}{\sum_{i=1}^n(X_i-\bar{X})^2} = \frac{\sum_{i=1}^n(X_i-\bar{X})^2+n(\bar{X}-\mu)^2}{\sum_{i=1}^n(X_i-\bar{X})^2} > t_\alpha$$

$$\frac{(\bar{X}-\mu)^2}{\sum_{i=1}^n(X_i-\bar{X})^2} > \tilde{t}_\alpha$$

$$T_n = \frac{\bar{X}-\mu}{\sqrt{\sum_{i=1}^n(X_i-\bar{X})^2}} > \gamma_\alpha$$

Let $X_i \sim \mathcal{N}(\mu,\sigma^2)$, $\bar{X} = \frac{\sum_{i=1}^n X_i}{n}$ be the sample mean, and $s^2 = \frac{\sum_{i=1}^n(X_i-\bar{X})^2}{n-1}$ the sample variance. Then $Z = \sqrt{n}\frac{\bar{X}-\mu}{\sigma} \sim \mathcal{N}(0,1)$, $U = (n-1)\frac{s^2}{\sigma^2} \sim \chi^2_{n-1}$, and the random variable $\tau = \frac{Z}{\sqrt{U/(n-1)}} \sim t_{n-1}$ follows a student's t law with $n-1$ degrees of freedom. Now $\tau = \frac{Z}{\sqrt{U/(n-1)}} = \sqrt{n}\frac{\bar{X}-\mu}{\sigma\sqrt{\sum_{i=1}^n(X_i-\bar{X})^2/\sigma^2}} = \frac{\sqrt{n}(\bar{X}-\mu)}{\sqrt{\sum_{i=1}^n(X_i-\bar{X})^2}}$ and therefore $\sqrt{n}T_n = \frac{\sqrt{n}(\bar{X}-\mu)}{\sqrt{\sum_{i=1}^n(X_i-\bar{X})^2}} \sim t_{n-1}$.

Exercise 4

1.(a)

$$\begin{aligned}
\text{reject } H_0 \text{ if } \quad & \frac{f(x_1,\cdots,x_n|H_1)}{f(x_1,\cdots,x_n|H_0)} > K_\alpha \\
& \frac{\prod_{i=1}^n \frac{\lambda_1^{x_i}}{x_i!}\exp(-\lambda_1)}{\prod_{i=1}^n \frac{\lambda_0^{x_i}}{x_i!}\exp(-\lambda_0)} > K_\alpha \\
& \frac{\lambda_1^{\sum_{i=1}^n x_i}}{\lambda_0^{\sum_{i=1}^n x_i}} - (\lambda_1-\lambda_0) > \ln K_\alpha \\
& \underbrace{(\ln\lambda_1 - \ln\lambda_0)}_{>0}\sum_{i=1}^n x_i > (\lambda_1-\lambda_0) + \ln K_\alpha \\
& T = \sum_{i=1}^n x_i > \frac{(\lambda_1-\lambda_0)+\ln K_\alpha}{\ln\lambda_1-\ln\lambda_0} = t_\alpha
\end{aligned}$$

with critical region $\Delta = \{(x_1,\cdots,x_n)|\sum_{i=1}^n x_i > t_\alpha\}$. *(Note that the sample mean could be used equivalently as the test statistic.)*

1.(b)

$$\begin{aligned}
\alpha &= P[\text{reject } H_0|H_0 \text{ true}] = P\left[T > t_\alpha|\lambda=\lambda_0\right] = P\left[T > t_\alpha|T\sim\mathcal{P}(n\lambda_0)\right] \\
&= 1 - P\left[T \le t_\alpha|T\sim\mathcal{P}(n\lambda_0)\right] = 1 - \sum_{k=0}^{\lfloor t_\alpha\rfloor}\frac{(n\lambda_0)^k}{k!}\exp(-n\lambda_0)
\end{aligned}$$

1.(c) We find that

k	0	1	2	3	4	5	6	7
$P[X>k]$	0.8647	0.5940	0.3233	0.1429	0.0527	0.0166	0.0045	0.0011

and therefore choose $t_\alpha \in (6,7]$, for instance $t_\alpha = 6.5$. The significance of the test is $\alpha = 0.0045$.

$$\begin{aligned}
\beta &= P[\text{reject } H_0|H_1 \text{ true}] = P\left[T > t_\alpha|\lambda=\lambda_1\right] = P\left[T > t_\alpha|T\sim\mathcal{P}(n\lambda_1)\right] \\
&= 1 - P\left[T \le t_\alpha|T\sim\mathcal{P}(n\lambda_1)\right] = 1 - \sum_{k=0}^{\lfloor t_\alpha\rfloor}\frac{(n\lambda_1)^k}{k!}\exp(-n\lambda_1).
\end{aligned}$$

For $\lambda = 2$, this gives $\pi = 0.3937$.

2.(a)

$$\begin{aligned}
\text{reject } H_0 \text{ if } \quad & \frac{f(x_1,\cdots,x_n|H_1)}{f(x_1,\cdots,x_n|H_0)} > \frac{P(H_0)}{P(H_1)}\frac{(c_{10}-c_{00})}{(c_{01}-c_{11})} \\
& \frac{\prod_{i=1}^n \frac{\lambda_1^{x_i}}{x_i!}\exp(-\lambda_1)}{\prod_{i=1}^n \frac{\lambda_0^{x_i}}{x_i!}\exp(-\lambda_0)} > \frac{P(H_0)}{P(H_1)}\frac{(c_{10}-c_{00})}{(c_{01}-c_{11})} \\
& \frac{\lambda_1^{\sum_{i=1}^n x_i}\exp(-\lambda_1)}{\lambda_0^{\sum_{i=1}^n x_i}\exp(-\lambda_0)} > \frac{P(H_0)}{P(H_1)}\frac{(c_{10}-c_{00})}{(c_{01}-c_{11})} \\
\underbrace{(\ln\lambda_1-\ln\lambda_0)}_{>0}\sum_{i=1}^n x_i - (\lambda_1-\lambda_0) & > \ln\left(\frac{P(H_0)}{P(H_1)}\frac{(c_{10}-c_{00})}{(c_{01}-c_{11})}\right) \\
T = \sum_{i=1}^n x_i & > \frac{1}{\ln\lambda_1-\ln\lambda_0}\left[(\lambda_1-\lambda_0) + \ln\left(\frac{P(H_0)}{P(H_1)}\frac{(c_{10}-c_{00})}{(c_{01}-c_{11})}\right)\right]
\end{aligned}$$

2.(b) The test simplifies to

$$T = \sum_{i=1}^n x_i > \frac{1}{\ln\lambda_1-\ln\lambda_0}\left[(\lambda_1-\lambda_0) + \ln\left(\frac{P(H_0)}{P(H_1)}\frac{c_{10}}{c_{01}}\right)\right]$$

and depends on the ration of c_{10} (cost for "false alarm") and c_{01} (cost for "non-detection"), which reflects the trade-off between false alarm and non detection probabilities.

3.(a) The χ^2 test is used since the random variables X_i are discrete and the Kolmogorov test therefore does not apply.
3.(b) The mean and variance of T are both given by λ, with $\lambda = n\lambda_0 = 3$. With the upper quartiles of the standard Normal distribution are given by $a = 0.6745$. this gives the $K = 4$ equi-probable classes

$$
\begin{aligned}
C_1 &= (-\infty,\, \lambda - \sqrt{\lambda}a] = (-\infty,\, 1.83] \\
C_2 &= (\lambda - \sqrt{\lambda}a,\, \lambda] = (1.83,\, 3] \\
C_3 &= (\lambda, \lambda + \sqrt{\lambda}a] = (3,\, 4.17) \\
C_4 &= (\lambda + \sqrt{\lambda}a,\, +\infty) = (4.17,\, \infty)
\end{aligned}
$$

The χ^2 test statistic is given by $\phi = \sum_{k=1}^{K} \left(\frac{(n_k - Np_k)^2}{Np_k} \right)$, where n_k are the number of observations per class, and p_k are the probabilities of the theoretical law for each class. Here, $Np_k = 30 \cdot 0.25 = 7.5$, $(n_1, n_2, n_3, n_4) = (4, 13, 4, 9)$ which yields $\phi = 7.6$. Under H_0, ϕ follows a χ^2_{K-1} law. The critical value is hence $t_{\alpha=0.1} = 6.25$, and the test rejects H_0 with significance $\alpha = 0.1$.
3.(c) Since $\lambda = n\lambda_0 = 20$, the limits of the $K = 4$ classes are now given by $(-\infty,\, 16.98,\, 20,\, 23.02\, \infty)$, $(n_1, n_2, n_3, n_4) = (7, 11, 6, 6)$ and $\phi = 2.27$. Hence, H_0 is accepted with significance $\alpha = 0.1$.

Exercise 5

1.(a)

$$\text{reject } H_0 \text{ if } \quad \frac{f(x_1,\cdots,x_n|H_1)}{f(x_1,\cdots,x_n|H_0)} > K_\alpha$$

$$\frac{\alpha_1^n x_m^{n\alpha_1}}{\left(\prod_{i=1}^n x_i\right)^{\alpha_1+1}} \frac{\left(\prod_{i=1}^n x_i\right)^{\alpha_0+1}}{\alpha_0^n x_m^{n\alpha_0}} > K_\alpha$$

$$\left(\frac{\alpha_1}{\alpha_0}\right)^n \frac{\left(\prod_{i=1}^n \frac{x_i}{x_m}\right)^{\alpha_0}}{\left(\prod_{i=1}^n \frac{x_i}{x_m}\right)^{\alpha_1}} > K_\alpha$$

$$\underbrace{(\alpha_0-\alpha_1)}_{<0}\sum_{i=1}^n \ln\left(\frac{x_i}{x_m}\right) + n\ln\left(\frac{\alpha_1}{\alpha_0}\right) > \ln K_\alpha$$

$$t = \sum_{i=1}^n \ln\left(\frac{x_i}{x_m}\right) < t_\alpha$$

The critical region of the test is $\Delta = \{(x_1,\cdots,x_n)| \sum_{i=1}^n \ln\left(\frac{x_i}{x_m}\right) < t_\alpha\}$.

1.(b)

$$\begin{aligned}
\alpha &= P[\text{reject } H_0|H_0 \text{ true}] = P\left[T < t_\alpha | T \sim \mathcal{G}\left(n, \frac{1}{\alpha_0}\right)\right] \\
&= \int_0^{t_\alpha} \frac{\alpha_0^n}{n!}\exp(-\alpha_0 u)u^{n-1}du \qquad | \text{ change of variable } v = \alpha_0 u \\
&= \int_0^{\alpha_0 t_\alpha} \frac{\alpha_0^n}{n!}\exp(-v)\frac{v^{n-1}}{\alpha_0^{n-1}}\frac{dv}{\alpha_0} \\
&= \frac{1}{n!}\int_0^{\alpha_0 t_\alpha} \exp(-v)v^{n-1}dv = I_n(\alpha_0 t_\alpha) \\
&\implies t_\alpha = \frac{1}{\alpha_0} I_n^{-1}(\alpha)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\pi(\alpha) &= P[\text{reject } H_0|H_1 \text{ true}] = P\left[T < t_\alpha | T \sim \mathcal{G}\left(n, \frac{1}{\alpha_1}\right)\right] \\
&= \int_0^{t_\alpha} \frac{\alpha_1^n}{n!}\exp(-\alpha_1 u)u^{n-1}du \\
&= \frac{1}{n!}\int_0^{\alpha_1 t_\alpha} \exp(-v)v^{n-1}dv = I_n(\alpha_1 t_\alpha) \\
&\implies \pi(\alpha) = I_n(\alpha_1 t_\alpha) = I_n\left(\frac{\alpha_1}{\alpha_0} I_n^{-1}(\alpha)\right),
\end{aligned}$$

where $\pi(\alpha)$ for α fixed only depends on $\frac{\alpha_1}{\alpha_0}$ (and n).

1.(c) We have that $I_n(z) = \frac{1}{n!}\int_0^z \exp(-v)v^{n-1}dv$ and $I_1(z) = \int_0^z \exp(-v)dv = 1-\exp(-z)$. From $I_1(\alpha_0 t_\alpha) = 1 - \exp(-\alpha_0 t_\alpha) = \alpha$ we obtain that

$$t_\alpha = \frac{1}{\alpha_0} I_1^{-1}(\alpha) = -\frac{\ln(1-\alpha)}{\alpha_0}.$$

Similarly, $\pi(\alpha) = I_1(\alpha_1 t_\alpha) = I_1\left(-\alpha_1/\alpha 0 \ln(1-\alpha)\right) = 1 - \exp\left(\frac{\alpha_1}{\alpha_0}\ln(1-\alpha)\right)$ and hence

$$\pi(\alpha) = 1 - (1-\alpha)^{\frac{\alpha_1}{\alpha_0}}.$$

2.(a)

$$F(x) = P[X \le x] = \begin{cases} 0 & \text{if } x < x_m \\ \int_{x_m}^x \frac{\alpha x_m^\alpha}{u^{\alpha+1}}du & \text{otherwise} \end{cases}$$

Hence, for $x > x_m$:

$$F(x) = \int_{x_m}^x \frac{\alpha x_m^\alpha}{u^{\alpha+1}}du = x_m^\alpha \int_{x_m}^x \alpha u^{-(\alpha+1)}du = x_m^\alpha(x_m^{-\alpha} - x^{-\alpha}) = 1 - \left(\frac{x}{x_m}\right)^{-\alpha}$$

2.(b)

$$F_0(x) = \left(1 - \frac{1}{x}\right) \mathbf{1}_{[1,+\infty]}$$

$$\text{reject } H_0 \text{ if } \quad \sup_{x\in\mathbb{R}} \left|F_0(x) - \hat{F}(x)\right| = \sup_{i\in\{1,2,3,4,5\}} \max\left(E_i^-, E_i^+\right) > t_\alpha$$

$$\text{where } E_i^- = \left|F_0(x_i^*) - \frac{i-1}{n}\right| \text{ and } E_i^+ = \left|F_0(x_i^*) - \frac{i}{n}\right|$$

i	1	2	3	4	5
x_i	2	3	5	6	7
F_0	1/2	2/3	4/5	5/6	6/7
E_i^+	|1/2-1/5|=3/10	|2/3-2/5|=4/15	|4/5-3/5|=1/5	|5/6-4/5|=1/30	|6/7-1|=1/7
E_i^-	1/2	|2/3-1/5|=7/15	|4/5-2/5|=2/5	|5/6-3/5|=7/30	|6/7-4/5|=2/35
$\max\left(E_i^-, E_i^+\right)$	1/2	7/15	2/5	7/30	1/7

Hence, $\sup_{x\in\mathbb{R}} \left|F_0(x) - \hat{F}(x)\right| = 1/2$, and H_0 is rejected with $\alpha = 0.1$, and accepted with $\alpha = 0.05$.