Sparse wave-packet representations and seismic imaging

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SUMMARY

In this paper we introduce a new algorithm for seismic imaging based on the flow out of Gaussian wave packets. We follow the standard strategy of decomposing data into wave packets and flowing them out along rays to approximate the downward wavefield extrapolation, and finally applying an imaging condition.

We revisit each computational step to gain efficiency. Furthermore, we develop procedure for seismic data decomposition in order to obtain highly sparse representations with Gaussian wave packets. As a result we get fast algorithm heavily exploiting sparse data representation and analytic description of Gaussian wave packets. We tests our algorithm on synthetic example of migrating common-shot gather.

INTRODUCTION

Gaussian beams were used for modeling seismic wave propagation since early 80s (Popov, 1982; Červený, 2001), while Gaussian wave packets introduced at the same time (Babich and Ulin, 1984) were not as popular in practical computations. The practical implementation of prestack migration with Gaussian beams was proposed by (Hill, 2001). Migration based on Gaussian packets was discussed in Bucha (2009); Žáček (2004); Klimeš (2004) with a detailed discussion of the related issues.

Recently, different versions of so called "beam migration" are have become popular in the seismic industry. The main strategy is to perform directional analysis of data and "steer" beams accordingly into the subsurface following the detected directions. It was noticed that curvelet frames can be used for sparse representation of data and migration operators (Douma and De Hoop, 2007; Chauris and Nguyen, 2008). In this paper we use Gaussian wave packets to represent seismic data, and use the fact that they are described by explicit analytic formulas that are invariant under many operations, e.g., translation, scaling, rotation, multiplication, convolution and Fourier transformation.

We address some new approaches to implementing the seismic migration operator based on the flow-out of Gaussian wave packets. Following Bucha (2009), we will discuss the main steps of the migration strategy for 2D common-shot gathers:

- data decomposition into Gaussian wave packets;
- flow-out of wave-packets into the subsurface;
- imaging condition (cross-correlation of wave packets).

THEORY

Wave-packet design

In two dimensions, we start from from the tensor product of two Gaussian, along with an oscillatory factor:

$$\varphi_{\alpha,\beta,\mathbf{k}}(\mathbf{x}) = e^{2\pi i \mathbf{k}^t \mathbf{x}}\, e^{-\mathbf{x}^t \mathbf{L}(\alpha,\beta,\mathbf{k})\mathbf{x}}, \tag{1}$$

where $\mathbf{x} = (x_1, x_2)^t$, $\mathbf{k} = (k_1, k_2)^t$, superscript t means transposition, and the matrix

$$\mathbf{L}(\alpha,\beta,\mathbf{k}) = \ln(16) \begin{pmatrix} \|\mathbf{k}\|^2/\alpha^2 & 0 \\ 0 & \|\mathbf{k}\|^2/\alpha^2\beta^2 \end{pmatrix} \tag{2}$$

describes the decay coefficients in two orthogonal directions.

In the next step, we apply rotations and translations to obtain the general Gaussian wave packet

$$\varphi_\gamma(\mathbf{x}) = e^{2\pi i \mathbf{k}^t (\mathbf{x}-\mathbf{y})} \exp\left(-(\mathbf{x}-\mathbf{y})^t \mathbf{R}_\theta^t \mathbf{L}(\alpha,\beta,\mathbf{k}) \mathbf{R}_\theta (\mathbf{x}-\mathbf{y})\right), \tag{3}$$

where $\mathbf{R}_\theta$ is the rotation matrix

$$\mathbf{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \tag{4}$$

and $\gamma = (\mathbf{y}, \mathbf{k}, \alpha, \beta, \theta)$ is the set of parameters. We use a collection of all wave packets $\gamma \in \Gamma$ in our construction of a discrete transform.

For the typical case, where $\mathbf{k} = |\mathbf{k}|(\cos\theta, \sin\theta)$, i.e., when the oscillation aligns with the semi-axis, $\mathbf{L}(\alpha,\beta,\mathbf{k})$ describes the decay in the direction along and orthogonal to the oscillation, respectively. Then a wave packet (1) can be viewed as "localized" plane wave as shown in Fig. 1; oscillatory direction becomes the one orthogonal to wave front and the other direction is tangent to the front (smoothly decaying).

To summarize, the parameters that define the Gaussian wave packets are:

- $\mathbf{y}$ - central location;
- $\mathbf{k}$ - vector of wavenumbers / scale;
- α - number of oscillations within a half-width;
- β - ratio of the half-widths along and perpendicular to the direction of oscillation;
- θ - orientation / rotation .

Data decomposition - sparse representation

Given a function (seismic data, image or snapshot) $f(\mathbf{x})$ and a collection of wave packets φ_γ, we want to find a set of coefficients $\mathbf{c} = \{c_\gamma\}$ such that

$$f(\mathbf{x}) \approx \mathrm{F}\mathbf{c} = \sum_\gamma c_\gamma \varphi_\gamma(\mathbf{x}), \tag{5}$$

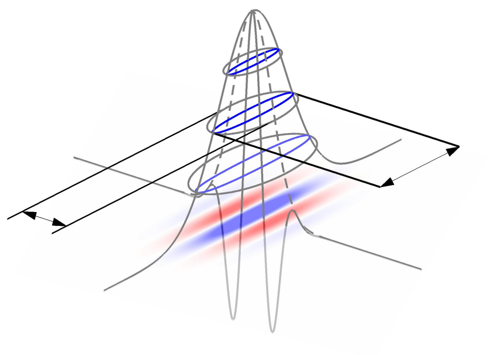

Figure 1: Gaussian wave packet can be viewed as "localized" plane wave.

where F is a recomposition operator, and we want only few of the elements c_γ, $\gamma \in \Gamma$, being non-zero. A popular way to obtain such a *sparse* representation is by solving the problems of the form

$$\min_{\mathbf{c}} \frac{1}{2}\|\mathrm{F}\mathbf{c} - f\|_2 + \mu\|\mathbf{c}\|_p, \qquad (6)$$

where $\|\cdot\|_p$ and $\|\cdot\|_2$ denote $\ell_p$and ℓ_2 norm, respectively.

The problem of developing computationally efficient algorithms to solve quadratic programming problems such as (6) for $p = 1$ has received much attention in recent years (Daubechies et al., 2004; Herrman et al., 2008). Main problem with these algorithms is that they are usually slowly converging and thus require many iterations (full image decomposition is required at each iteration).

We have developed and implemented (in 2D) a new algorithm for solving non-linear problems of the form (6) for $p \leq 1$. Main novelty is that we use very fast internal iterations with only a few full decompositions of an image (cf. (Andersson et al., 2010)).

So far we have implemented a 2D decomposition that can be used for getting sparse representation of 2D seismic data sets with rather few wave packets. Sparse data representations are important for reducing the computational cost for the whole migration procedure, i.e., reducing the number of rays to be traced and the cost of applying imaging condition. In addition, we essentially get a high quality analysis of data directionality that can be used in many different ways apart from migration: detecting slopes, data regularization, as a part of event picking etc.

Flow-out of wave packets

Wave-packet flow-out is the most restrictive step of our migration strategy. It was noticed by many authors that a smooth migration velocity model should be used for propagation (flow-out) of Gaussian wave packets. Even using a smooth model one gets in trouble trying to propagate Gaussian packets for a long distance. For spreading waves, holes start to appear between propagating wave packets. For focusing waves one can get incorrect interference pattern from wave packets 'overriding' each other.

For now, we restrict ourselves to the so called rigid flow of wave packets: each packet is moved along a ray and stretched in the ray direction according to velocity at the packet central point. Although not ideal, it is illustrative because of its simplicity, and it easily allows for fast flow-out implementations. We use rigid flow to calculate propagation of 'source' and 'receiver' wavefields downwards into subsurface.

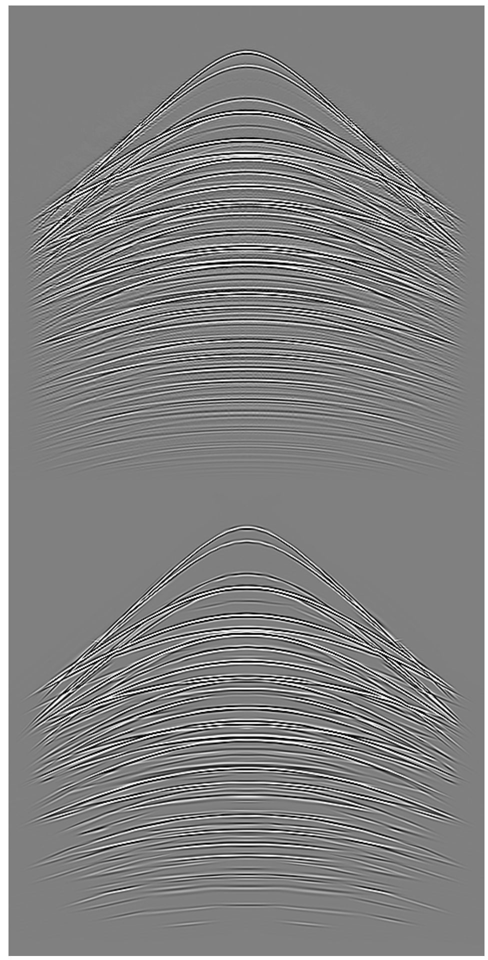

Figure 2: Wave-packet decomposition of synthetic data. Top: original data; bottom: representation with 1129 wave packets.

Imaging condition

While restricting ourselves to a rigid flow of wave packets we notice that Gaussian wave-packets maintain their analytic description in coarse of propagation. Thus we can write out analytic formula for applying imaging condition. Parameters of two Gaussian wave packets (one from the 'source' and one from the 'receiver' fields) are recalculated into parameters of another Gaussian wave packet that corresponds to their zero-lag cross-correlation assuming that local homogeneity of a medium.

EXAMPLES

Sparse representation of data and snapshots

In this subsection we apply our Gaussian wave packet decomposition method for sparse representation of synthetic shot gather that contains many overlapping events. We show original data set in Fig. 2, top. In Fig. 2, bottom we show sparse representation of this data with 1129 wave packets. It took about 7000 iterations to solve the problem (6) for this image including only 7 full decompositions. Total decomposition time was around 2 minutes on typical quad-core desktop.

Migration for a common-shot gather

In this subsection we test our migration algorithm on a simple synthetic example. We take one shot gather containing one wave reflected from a horizontal boundary at depth $z = .75$ km; upper layer is homogeneous with velocity $v_0 = 1.5$ km/s. Shot gather was generated by a standard FD method (see Fig. 3, top) and then decomposed into 26 Gaussian wave packets (see Fig. 3, bottom). One can see that data can be pretty well represented by these 26 wave packets. We also use data representation with 41 wave packets that provides almost exact data recovery (not shown).

These wave packets were transformed from the data (space-time) domain into the subsurface domain. From the parameters of the transformed wave packets we can find initial data and trace all 26 (or 41) rays (backward in time) describing downward continuation of the 'receiver' field. We also trace 20 rays from the source location (forward in time) with uniformly distributed initial angle. These rays are shown in Fig. 4, top. Black lines correspond to 'source' rays while blue lines correspond to 'receiver' rays.

We then define the typical wave-packet width for each ray (it can be defined varying along the ray in case of heterogeneous velocity model). While stepping along rays with typical half-period of wave packets, we check the closest distance between all source and receiver rays. This procedure can become computationally expensive when there are many rays involved. In our implementation we rely on the assumption that we can get a sparse representation of data. Thus we assume that number of 'receiver' rays is always small.

Finally for each pair of wave packets chosen we apply analytic formula implementing imaging condition and get back a wave packet that is a part of an image. In Fig. 5 we show final image that is a stack of all image wave packets. Fig. 5, top corresponds to 21 data wave packets that resulted into 47 image wave packets. It means that each 'receiver' wave packet intersected with less than two 'source' wave packets. Fig. 5, bottom corresponds to the the case of 41 data wave packets. It is composed of 117 image wave packets. Thus in this case each 'receiver' wave packet intersected with almost three 'source' wave packets. One can see the importance of sparse representation: with increasing number of wave packets used complexity of the algorithm is growing faster then linearly.

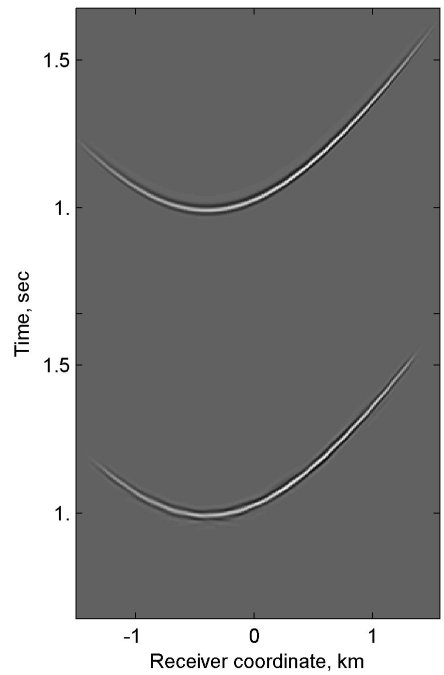

Figure 3: Original shot gather (top) and its representation with 26 packets (bottom).

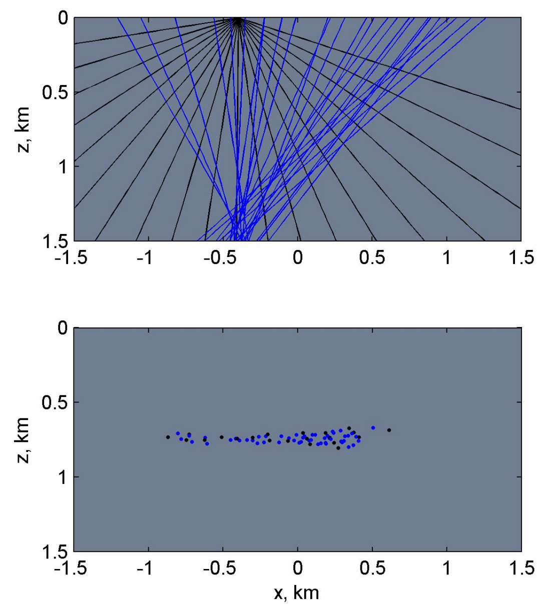

Figure 4: Kinematics of wave-packet flow-out. Top: 'source' rays (black) and 'receiver' rays (blue); bottom: pairs of points where 'source' wave packets (corresponding to the black dots) come close by 'receiver' wave packets (blue dots).

DISCUSSION

We note that images in Fig. 5 almost perfectly align at correct depth of the reflector, $z = .75$ km. There are no artefacts that will be typical for Kirchhoff migration applied to one shot gather.

One can see that decomposing data into 41 wave packets is resulting in a better and more continuous image compared to using 26 wave packets. Also one can see that image is a bit stretching at the right side. This part of the image corresponds to very steep events in data close to aliased region. More careful conversion of wave packets from data to subsurface domain should be used in this case. Also note that this region is close to critical reflections that may also affect quality of imaging.

The most crucial step in the migration procedure is the downward extrapolation of ‘receiver’ and ‘source’ field. Here we implements it as a rigid flow of Gaussian wave packets along rays: their translation and rotation defined by ray geometry. There are two reasons for this choice: it is fast and wave packets during the flow-out are still described by analytic formula (3).

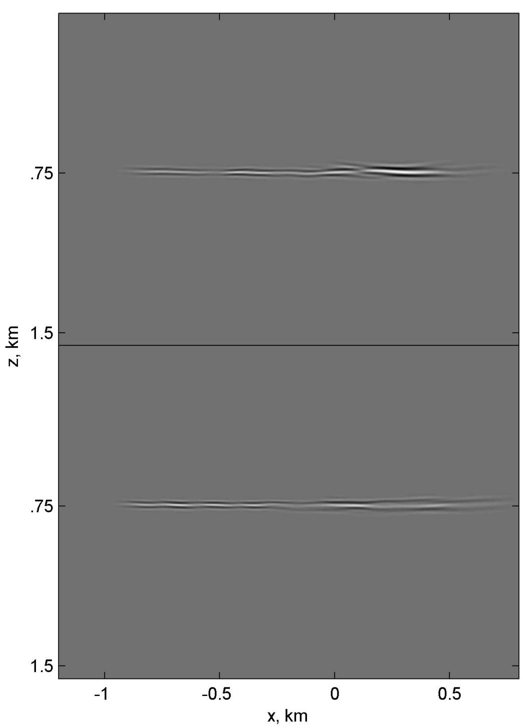

Figure 5: Migrated image. Top: with data decomposed into 21 wave packets; bottom: with data decomposed into 41 wave packets.

One drawback of this approach is that we implicitly need to make an assumption about a smooth migration velocity model used. However, note that this limiting assumption is typical for all methods based on wave packet flow-out (cf. Bucha (2009)).

This procedure can become computationally expensive when there are many rays involved. In our implementation we rely on the assumption that we can get a sparse representation of data. Thus we assume that number of ‘receiver’ rays is always small.

Note also that our decomposition algorithm was not designed just as a building block of our migration method. It provides sparse representation of ‘wave-type’ objects: seismic data, image or snapshots. It can be useful in many fields of data processing: directionality analysis, data regularization, event picking etc. This can be illustrated if we draw central points and orientations for the data representation example from Fig. 2. Orientations are sown in Fig. 6 with arrows starting from central points of corresponding wave packets.

CONCLUSIONS

We have introduced and tested on synthetic data a new algorithm for common-shot gather migration. This algorithm is based on a flow-out of Gaussian wave packets. Data is first decomposed into wave packets. These wave packets are further moved into subsurface along corresponding rays while remaining Gaussian wave packets. Then analytic formula is used for fast implementation of imaging conditions.

In conclusion we repeat that imaging with wave packets has natural restriction, i.e., it can be used only for smooth models. However it has all the flexibility of Kirchhoff migration. Thus we think that it can be competitive as a tool for preliminary, iterative or target-oriented migration given that its implementation is computationally cheap.

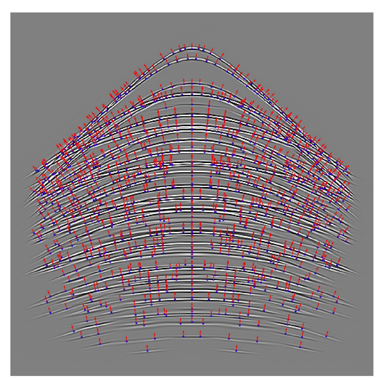

Figure 6: Central points and orientations (arrows) of 1129 wave packets used to represent data in Fig. 2.

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