A Characterization of Generalized Concordance Rules in Multicriteria Decision Making

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This article proposes a principled approach to multicriteria decision making (MCDM) where the worth of decisions along attributes is not supposed to be quantified, as in multiattribute utility theory, or even measured on a unique scale. This approach actually generalizes additive concordance rules *a la Electre* and is rigorously justified in an axiomatic way by representation theorems. We indeed show that the use of a generalized concordance (GC) rule is the only possible approach when in a purely ordinal framework and that the satisfaction of very simple principles forces the use of possibility theory as the unique way of expressing the importance of coalitions of criteria. © 2003 Wiley Periodicals, Inc.

1. INTRODUCTION

There are at least three decision-making problems that have been studied rather independently in the past: individual decision making under uncertainty (DMU), multiagent decision making (MADM), and multicriteria decision making (MCDM).

DMU research has culminated with the works of Savage[1] and advocates a numerical approach to decision making, whereby uncertainty is represented by a single probability function, preference is encoded by utility functions, and acts are ranked according to expected utility. It has been extended, among others, by Schmeidler,[2] and Sarin and Wakker[3] on the basis of the Choquet integral. MADM is, for a large part, based on the works of Arrow[4] who formulated it in a qualitative

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framework as a voting problem in which complete preordering relations must be aggregated. It leads to an impossibility theorem.

However, DMU and MADM are similar problems if the set of states of the world is viewed as the set of voters in MADM.[5] It may be surprizing that, formally, similar problems lead to different settings, one being quantitative and the other qualitative. The axiomatic framework of Savage has been reconsidered in the scope of qualitative decision theory[6,7] and it has been shown that keeping the essential features of Savage's framework while sticking to an ordinal framework leads either to an impossibility theorem or to decision rules that generalize Condorcet's pairwise relative majority rule.

In this study, we focus on the third problem, namely, MCDM which is also of the same structure as MADM and DMU, the set of criteria playing the role of voters (respectively, of states). Interestingly, here, there are two distinct schools of thought in this area, one deriving from the DMU tradition, and the other from voting theory. The first school is essentially numerical and puts forward a weighted sum for the aggregation of scaled utility functions; several principled justifications have been proposed, principally in conjoint measurement, dealing with additive utility.[8–11] The second school stems from the works of Roy[12]: preferences along each criteria (whether numerical or not) are represented by an outranking relation. So-called additive concordance rules are used to perform criteria aggregation; they are based on counting the number of criteria that favor one alternative over another.

There are actually very few foundational works in ordinal MCDM in comparison with multicriteria numerical utility theory. However, one has to mention the pioneering work of Fishburn[13] (see also Refs. 14 and 15 and more recent works linked to the application of nontransitive conjoint measurement to MCDM, e.g., Ref. 16). Moreover, the problem of deriving aggregation procedures compatible with a qualitative approach has seldom been considered if we except some works in artificial intelligence, namely, in information fusion.[17–19] The aim of this study is to lay bare natural axioms that a purely ordinal approach to MCDM should intuitively satisfy. These axioms characterize generalized concordance (GC) rules, whose practical use is very limited; this work points out the limitation of such ordinal approaches. It parallels the one started in Ref. 6, taking advantage of the similarities between MCDM and DMU (however, the framework adopted here is more general). In the next Section we present the two approaches to MCDM in more detail and discuss the strong assumptions underlying the numerical aggregation scheme. Section 3 then proposes a generalization of the usual additive concordance rule. Finally, Section 4 proposes a rigorous axiomatization of such generalized decision rules. For the sake of clarity, proofs are given in Appendix A.

2. PRELIMINARY DEFINITIONS AND REMARKS

A multicriteria decision problem can be characterized by a set $\mathcal{A}$ of *alternatives* (possible actions, objects, and candidates) and a set $N = \{1, \ldots, n\}$ of *attributes* or *criteria* used to describe the alternatives. Let X_j denote the set of possible values for component $j \in N$ and $X = X_1 \times \cdots \times X_n$ be the *multiattribute space.* Within X, each alternative $\mathbf{x} \in \mathcal{A}$ is represented by the vector

$(x_1, \ldots, x_n)$ of attributes values. $\mathcal{A}$ can be identified to its image in X and thus considered as a subset of X. $\mathcal{A}$ will actually be identified with X itself because we need a comparison model allowing the decision of whether $\mathbf{x}$ is at least as good as $\mathbf{y}$ (denoted $\mathbf{x} \succcurlyeq \mathbf{y}$) or not, *whatever* $(\mathbf{x}, \mathbf{y})$ in X^2, i.e., a comparison model well defined on the entire set X.

As a first consequence, for any pair $(\mathbf{x}, \mathbf{y})$ of alternatives and for any subset of attributes $A \subset N$, we can soundly construct a mixed alternative $\mathbf{x}A\mathbf{y}$ in which its components are those of $\mathbf{x}$ on the elements of A and those of $\mathbf{y}$ on the other attributes:

$$(\mathbf{x}A\mathbf{y})_j = \begin{cases} x_j & \text{if } j \in A \\ y_j & \text{if } j \notin A \end{cases} \quad j = 1, \ldots, n$$

More generally, $x_1A_1x_2A_2 \ldots x_kA_kz$ is the alternative whose components are those of x_i on the elements of A_i and those of z are on the other elements.

Each attribute j usually defines a marginal utility $u_j(x_j)$ measuring the attractiveness of the attribute value x_j. We assume that $u_j(x_j) \in [0, 1]$ but any linear scale could be considered as well. In some models, the scale is ordinal in nature and the marginal utility only encodes a ranking of the set X_j. In this study, both $\mathcal{A}$ and N are supposed to be finite. Consistently, each X_j also admits a finite set of values.

MCDM and MADM methods often involve a measure μ on 2^N that represents the level of importance of the coalitions (of criteria or of voters). This importance measure must be a *capacity* on N, i.e., a mapping defined from 2^N to $[0, 1]$ such that $\mu(\emptyset) = 0$, $\mu(N) = 1$, and $\mu(A) \leqslant \mu(B)$ for any pair of subsets (A, B) in N such that $A \subseteq B$. Important subclasses of capacities are formed by

- Additive capacities (e.g., probabilities) characterized by

$$\forall\, A \subseteq N, \quad \mu(A) = \sum_{j \in A} \mu(\{j\})$$

  Such measures are autodual and for any additive capacity there obviously exists a distribution $p : N \mapsto [0, 1]$ such that for all $A \subseteq N$, $\mu(A) = \Sigma_{j \in A}\, p(j)$.
- Possibility measures characterized by

$$\forall\, A, B \subseteq N, \quad \mu(A \cup B) = \max(\mu(A), \mu(B))$$

  Notice that for any possibility measure μ there exists a possibility distribution $\pi : N \mapsto [0, 1]$ such that for all $A \subseteq N$, $\mu(A) = \max_{j \in A} \pi(j)$.
- Necessity measures characterized by

$$\forall\, A, \quad B \subseteq N, \quad \mu(A \cap B) = \min(\mu(A), \mu(B))$$

  Notice that for any necessity measure μ, there exists a possibility distribution $\pi : N \mapsto [0, 1]$ such that for all $A \subseteq N$, $\mu(A) = 1 - \max_{a \in \bar{A}} \pi(a)$. This is because of the fact the dual of a possibility measure is always a necessity measure (and conversely).

Finally, the preference relation $\succcurlyeq$ on X^2 is built through using a *decision rule* defining the preference $\mathbf{x} \succcurlyeq \mathbf{y}$ as a function of vectors $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$. As said previously, one can distinguish between two approaches to derive $\succcurlyeq$.

2.1. The Aggregate and then Compare Approach

This approach consists of summarizing each vector $\mathbf{x}$ by a quantity $u(\mathbf{x})$. This utility is obtained by aggregation of marginal utilities $u_j(x_j)$, often by means of a weighted sum. More formally, the preference relation $\succcurlyeq$ is defined by

$$\mathbf{x} \succcurlyeq \mathbf{y} \Leftrightarrow \phi(\psi((u_1(x_1), \ldots, u_n(x_n))), \psi((u_1(y_1), \ldots, u_n(y_n))) \geqslant 0$$

where ψ denotes the aggregation operator and ϕ is a comparison function. A classical choice for ϕ is $\phi(\alpha, \beta) = \alpha - \beta$ but more sophisticated models using discrimination thresholds could be used (see, e.g., Ref. 20). We assume here that ψ is defined from $[0, 1]^n$ to $[0, 1]$ and such that $\psi(0, \ldots, 0) = 0$ and $\psi(1, \ldots, 1) = 1$. For example, we can choose for ψ a particular instance of Choquet or Sugeno integrals defined by

$$C_\mu(\alpha_1, \ldots, \alpha_n) = \sum_{j=1}^{n} (\alpha_{(j)} - \alpha_{(j-1)})\mu(\{(j), \ldots, (n)\})$$

$$S_\mu(\alpha_1, \ldots, \alpha_n) = \bigvee_{j=1}^{n} \alpha_{(j)} \wedge \mu(\{(j), \ldots, (n)\})$$

where μ is a capacity measure and $\alpha_{(j)}$, $j = 1, \ldots, n$ are the components of α ranked in increasing order:

$$\alpha_{(1)} \leqslant \alpha_{(2)} \leqslant \cdots \leqslant \alpha_{(n)}$$

and $\alpha_{(0)} = 0$. Choquet integral is a powerful aggregation operator allowing positive and negative synergies between criteria.[21,22] When used with an additive capacity, it boils down to the weighted sum. Sugeno integral is a qualitative counterpart of the Choquet integral.[23] When used with a possibility (respectively a necessity) measure, it boils down to a weighted max (respectively a weighted min).[24]

2.2. The Compare and then Aggregate Approach

This approach consists of comparing, for any pair $(\mathbf{x}, \mathbf{y})$ in X^2 and each attribute j in N, the consequences x_j and y_j so in order to decide whether $\mathbf{x}$ is at least as good as $\mathbf{y}$ according to the jth attribute. This yields n preference indices $\phi_j(\mathbf{x}, \mathbf{y})$, j $\in$ N. These preference indices restricted to a single attribute then are aggregated before performing the following preference test:

$$\mathbf{x} \succcurlyeq \mathbf{y} \Leftrightarrow \psi(\phi_1(\mathbf{x}, \mathbf{y}), \ldots, \phi_n(\mathbf{x}, \mathbf{y})) \geqslant \psi(\phi_1(\mathbf{x}, \mathbf{y}), \ldots, \phi_n(\mathbf{x}, \mathbf{y}))$$

Example 1 (Additive Concordance Rule). Let us define ϕ as

$$\phi_j(\mathbf{x}, \mathbf{y}) = \begin{cases} 1 & \text{if } u_j(\mathbf{x}) \geqslant u_j(\mathbf{y}) \\ 0 & \text{if } u_j(\mathbf{x}) < u_j(\mathbf{y}) \end{cases}$$

Choosing $\psi(\alpha_1, \ldots, \alpha_n) = \sum_{j=1}^{n} w_j \alpha_j$ leads to the weighted relative majority rule. When w_j are all equal, we obtain a well-known MADM system where $\mathbf{x} \succcurlyeq \mathbf{y}$ iff a majority of voters/attributes is concordant with this preference. In MCDM, interpreting the w_j as the weights of criteria leads to a concordance rule used in Electre methods.[25,26]

In this example, the integrals C_μ or S_μ could be used for ψ, thus leading to more sophisticated decision rules using the relative importance of the attributes.

Generally, many ordinal decision rules can be obtained as particular instances of the general comparison model compare and then aggregate (CA). This is the case of many voting procedures considered in social choice,[27–29] but also the case of concordance rules used in Electre methods for multicriteria decision analysis.[26,27,30,31]

The fact that the CA approach covers ordinal decision rules should be noted. The choice of ϕ_j in Example 1 indeed amounts to constructing n preference *relations,* which are aggregated by some ψ to form the overall preference *relation* $\succcurlyeq$.

2.3. On the Commensurability Between Local Preference Scales and Importance Scales

Both methods AC and CA use n preference scales $(L_j, \geqslant)$, each being characterized by a set of levels $L_j = \{u_j(x_j), x_j \in X_j\}$ ordered by $\geqslant$. The two approaches also require an aggregation operation ψ. When ψ explicitly uses an importance measure μ on N (this is the case of Sugeno and Choquet integrals) the aggregation operator can be denoted ψ_μ. In the AC approach, ψ_μ is used to aggregate marginal utility indices $u_j(x_j)$, whereas in CA ψ_μ is used to aggregate marginal preference indices $\phi_j(\mathbf{x}, \mathbf{y})$, resulting from pairwise comparisons. There is a specific scale L_μ for criteria importance levels (the range of μ, i.e., $L_\mu = \{\mu(A), A \subseteq N\}$). Hence, there are two distinct commensurability problems:

1. Can the same totally ordered preference scale L_u be attached to all criteria?
2. Is there a mapping relating levels of importance in L_μ and levels of satisfaction in L_j, $j = 1, \ldots, n$?

In the AC approach an affirmative answer is given to both questions. Indeed, one commonly assumes that the same utility scale $(L_u, \geqslant)$ is valid for each attribute. A possible choice for this scale is $L_u = \cup_{j \in N} L_j$. This choice requires the comparability of utility levels coming from different scales L_j, which is a strong assumption. On top of this single utility scale, we also need the importance scale L_μ. The conjoint use of scales L_u and L_μ in the definition of preferences implicitly makes the two scales commensurate.

To explain how this commensurability between importance and utility is achieved, consider four alternatives in X defined by

- The ideal alternative $\mathbf{x}^*$ such that $u_j(\mathbf{x}^*) = 1$, for all $j \in N$
- The anti-ideal alternative $\mathbf{x}_*$ such that $u_j(\mathbf{x}_*) = 0$, for all $j \in N$
- An alternative $\mathbf{a}$ having a constant utility vector $(\alpha, \ldots, \alpha)$, for $\alpha \in L_u$
- The alternative $\mathbf{x}^*A\mathbf{x}_*$ for a given proper subset A in N ($A \neq \emptyset$ and $N \backslash A \neq \emptyset$).

Most aggregation functions ψ_μ satisfy the two following properties[a]:

Idempotency for all $\alpha \in [0, 1]$, $\psi_\mu(\alpha, \ldots, \alpha) = \alpha$
Coincidence for all $A \subseteq N$, $\psi_\mu(\mathbf{x}^*A\mathbf{x}_*) = \mu(A)$

Following AC we can define the preference order $\succcurlyeq_\mu$ corresponding to ψ_μ by

$$\mathbf{x} \succcurlyeq_\mu \mathbf{y} \Leftrightarrow \psi_\mu(u_1(x_1), \ldots, u_n(x_n)) \geqslant \psi_\mu(u_1(y_1), \ldots, u_n(y_n))$$

In this context, it can be shown that

PROPOSITION 1. *If ψ_μ is idempotent and coincident, then for any proper subset A in N, we have*

$$(\mathbf{x}^*A\mathbf{x}_*) \succcurlyeq_\mu \mathbf{a} \Leftrightarrow \mu(A) \geqslant \alpha$$

This result reveals an implicit comparison between the level $\mu(A)$ in the scale L_μ and level α in the scale L_u. This shows that in the CA approach, the intermingling of the scales L_μ and L_u is instrumental in the comparison of alternatives, i.e., that L_μ and L_u need to be commensurate.

Recall that the similarity of the DMU, MADM, and MCDM decision frameworks lies in the fact that the set of attributes plays the same role as the set of states and the set of voters. The AC approach, accepting the two commensurability assumptions appears natural in DMU; the set of consequences of acts is indeed often independent of the considered state, and because there is a single decision-maker, there is a single preference scale for the consequences. The second commensurability problem is a matter of comparing degrees of uncertainty of events (the counterpart of degrees of importance) and degrees of preference of consequences. Although clearly distinct notions, uncertainty and preference are equated in DMU provided that the decision maker is able to compare a sure gain and a binary lottery (which characterizes an event; see Footnote a).

In the MADM context, the CA approach is much more natural because such commensurability assumptions are difficult to accept. Indeed, local preference scales L_j are attached to distinct voters and hence are hard to reconcile. Moreover, the importance of individual voters or groups thereof generally is determined by an external agent, not the voters themselves; hence, the commensurability between individual preference scales and the importance scale is not warranted.

[a]In DMU, idempotency holds when a constant act is equated to its unique consequence (a sure gain), and coincidence means that the confidence of event A is the utility of a binary act having extremely good consequences if A occurs and extremely bad ones if not, thus pointing out the commensurability hypothesis between uncertainty and preference scales.

In MCDM, the presence of a single decision maker makes the AC approach more natural than in the MADM setting, but it raises an important operational question. Indeed, to capture the preferences of decision makers in the AC model, it would be necessary to ask a huge number of questions aiming at defining exactly how elements of the various utility and importance scales should be intermingled on a common scale. We will show in the next section that concordance methods that are particular instances of the CA approach do not face this problem. From this point of view the MCDM problem appears to be closer to the MADM setting than to DMU, and it is natural to try and tackle the MCDM problem, making as few commensurability assumptions as possible.

3. GENERALIZED CONCORDANCE RULES

The additive concordance rules introduced in Section 2.2 can be cast in a more general setting. First, a preference relation $\succcurlyeq_j$ is supposed to exist on each attribute range X_j. It can be derived from the marginal utility functions if any [then $x_j \succcurlyeq_j y_j \Leftrightarrow u_j(x_j) \geq u_j(y_j)$] or introduced as such from scratch by the decision maker. Let $\succ_j$ and $\sim_j$ denote the strict preference and the indifference relations derived from $\succcurlyeq_j$. The following coalition of attributes derives from the marginal preferences:

$$C_{\succcurlyeq}(\mathbf{x}, \mathbf{y}) = \{j \in N, x_j \succcurlyeq_j y_j\}$$

$C_{\succcurlyeq}(\mathbf{x}, \mathbf{y})$ is the set of criteria where $\mathbf{x}$ is as least as good as $\mathbf{y}$.

Finally, assume an importance relation $\succcurlyeq_I$ exists on 2^N, whereby $A \succcurlyeq_I B$ means that the group of attributes A is as least as important as the group B. It can be derived from the importance function, if any, [then $A \succcurlyeq_I B \Leftrightarrow \mu(A) \geq \mu(B)$] or introduced as such from scratch by the decision maker. Such a relation is supposed to be reflexive and monotonic, i.e.,

$$A \succcurlyeq_I B \Rightarrow A \cup C \succcurlyeq_I B \quad \text{and} \quad A \succcurlyeq_I B \cup C \Rightarrow A \succcurlyeq_I B$$

This property is satisfied if $\succcurlyeq_I$ derives from a capacity as introduced in Section 2.1. It implies the usual monotony condition of capacity functions. Indeed, $A \succcurlyeq_I A$ implies $A \cup B \succcurlyeq_I A$ (i.e., for all A, C such that $A \subseteq C$, $C \succcurlyeq_I A$). The converse is not true, except when $\succcurlyeq_I$ is supposed to be transitive, as with capacity functions.

Importance relations derived from additive capacities also obey the following property of *preadditivity:*

$$\forall A, B, C \subseteq N \quad A \cap (B \cup C) = \varnothing \Rightarrow (B \succcurlyeq_I C \Leftrightarrow A \cup B \succcurlyeq_I A \cup C)$$

However, it is well known that the preadditivity of $\succcurlyeq_I$ does not imply its additivity (see the counterexample of Ref. 32, where a preadditive relation is exhibited that is not representable by an additive capacity).

Now, let us define generalized concordance (GC) rules.

DEFINITION 1 (GC Rules). *A* generalized concordance (GC) rule *defines a preference relation* $\succcurlyeq$ *on* X *from the local preference relations* $\succcurlyeq_j$ *on* X_j, *for all* $j = 1, \ldots, n$ *and the importance relation* $\succcurlyeq_I$ *on* 2^N *as follows:*

$$\mathbf{x} \succsim \mathbf{y} \Leftrightarrow C_{\succsim}(\mathbf{x}, \mathbf{y}) \succsim_I C_{\succsim}(\mathbf{y}, \mathbf{x})$$

This definition is an MCDM counterpart to (and a generalization of) the "lifting rule" proposed by Ref. 6 for DMU. When $\succsim_I$ (resp. $\succsim_j$) derives from a capacity function μ (resp. a utility function u_j) or equivalently when they are weak orders (and thus always representable by capacity and utility functions), the previous rule becomes

DEFINITION 2 (GC_μ Rules). *A* capacity-based concordance rule *defines a preference relation $\succsim$ on X from the relations $\succsim_j$ on X_j, for all $j = 1, \ldots, n$ and a capacity μ on 2^N as follows:*

$$\mathbf{x} \succsim \mathbf{y} \Leftrightarrow \mu(C_{\succsim}(\mathbf{x}, \mathbf{y})) \geq \mu(C_{\succsim}(\mathbf{y}, \mathbf{x}))$$

The additive concordance rule of Section 2.2 is recovered when μ is an additive capacity. When μ is a necessity function, one recovers the (necessity-based) concordance rule proposed by Ref. 6 in the context of DMU, namely,

$$\mathbf{x} \succsim \mathbf{y} \Leftrightarrow \mathrm{Nec}(C_{\succsim}(\mathbf{x}, \mathbf{y})) \geq \mathrm{Nec}(C_{\succsim}(\mathbf{y}, \mathbf{x}))$$

Remark that

- Variants of the GC rule can be obtained using $\succ_j$ (resp. $\succ_I$) instead of $\succsim_j$ (resp. $\succsim_I$) in Definition 1.
- Applied to alternatives $\mathbf{x}^*A\mathbf{x}_*$ and a introduced in Section 2.3, it is clear that $\mathbf{x}^*A\mathbf{x}_* \succsim \mathbf{a} \Leftrightarrow \mathbf{A} \succsim_I \mathbf{N}\backslash\mathbf{A}$ because $C_{\succsim}(\mathbf{x}^*A\mathbf{x}_*, \mathbf{a}) = A$ and $C_{\succsim}(\mathbf{a}, \mathbf{x}^*A\mathbf{x}_*) = N\backslash A$. One can observe that the result only depends on the inequality $A \succsim_I N\backslash A$, which pertains to two levels of the same scale. Generally, the GC rule does not require any commensurability assumption. Only comparisons within X_j and comparisons between sets of attributes are requested. Hence, when weak orders are considered (Rule GC_μ), changing the intertwining of quantities of type $\mu(A)$ with quantities $u_j(\mathbf{x})$ does not affect the preference $\succsim$ induced. Thus, assessing utility functions and the capacity μ is enough with this model contrary to the AC approach.
- When the relations $\succsim_j$ are complete, $C_{\succsim}(\mathbf{x}, \mathbf{y}) \cup C_{\succsim}(\mathbf{y}, \mathbf{x}) = N$. In this case, only a subpart of $\succsim_I$ is used, namely, the one that compares sets, the disjunction of which form the full attribute set.
- It is a priori natural to assume that $\succsim_I$ and the $\succsim_j$ are complete and transitive. However, the GC rule makes sense even if these properties do not hold. The transitivity of the relations $\succsim_j$ may be questioned; e.g., consider a numerical attribute that is naturally ordered (e.g., a price). It may happen that a "small variation" of a value on this attribute will not modify the subjective value of the alternative considered. Indeed, one can imagine that the decision maker remains indifferent between two values x_j and y_j as long as the difference $|x_j - y_j|$ does not exceed a certain threshold. Such preferences are perfectly natural but fail to be transitive. A similar rationale could be developed concerning the transitivity of relation $\succsim_I$.
- GC rules fit CA approach of Section 2.2 when $\succsim_I$ (resp. $\succsim_j$) derives from a capacity function μ (resp. a utility function u_j) or equivalently when they are weak orders (Rule GC_μ). The general CA scheme is recovered when ϕ_j is defined as in Section 2.2 and ψ is such that for all $\alpha \in [0, 1]^n$, $\psi(\alpha) = \mu(\boldsymbol{\sigma}(\alpha))$, where $\boldsymbol{\sigma}(\alpha)$ is a vector with component $\sigma_j(\alpha) = 1$ if $\alpha_j \geq 0$ and 0 otherwise. The term $\boldsymbol{\sigma}(\alpha)$ is the characteristic vector of $C_{\succsim}(\mathbf{x}, \mathbf{y})$ and $\boldsymbol{\sigma}(-\alpha)$ of $C_{\succsim}(\mathbf{y}, \mathbf{x})$.

A natural question is whether it is worth considering a nonadditive importance function μ to define GC rules. The claim that additive capacities are not expressive enough is based on counterexamples like the following.

Example 2. We evaluate and compare four candidates applying to a commercial engineering position. Candidates receive grades according to four points of view: technical skill ($X_1 = \{A, B, C, D, E\}$), commercial skill ($X_2 = \{A, B, C\}$), age ($X_3 = \{20, \ldots, 60\}$), and salary ($X_4 = \{20, \ldots, 100\}$). Within X_1 (resp. X_2), A is the best grade and C is the worst. Numerical values in X_3 and X_4 are to be minimized. The ratings of the four candidates $\{\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w}\}$ are

candid./attrib.	1	2	3	4
x	*B*	*B*	*31*	*60*
y	*C*	*A*	*31*	*60*
z	*B*	*B*	*49*	*50*
w	*A*	*C*	*26*	*80*

We get

$$C_{\geqslant}(\mathbf{x}, \mathbf{y}) = \{1, 3, 4\} \quad C_{\geqslant}(\mathbf{y}, \mathbf{x}) = \{2, 3, 4\}$$

$$C_{\geqslant}(\mathbf{x}, \mathbf{z}) = \{1, 2, 3\} \quad C_{\geqslant}(\mathbf{z}, \mathbf{x}) = \{1, 2, 4\}$$

$$C_{\geqslant}(\mathbf{x}, \mathbf{w}) = \{2, 4\} \quad C_{\geqslant}(\mathbf{w}, \mathbf{x}) = \{1, 3\}$$

Now assume that the decision maker's choice is $\mathbf{x}$ (a reasonable choice because this candidate realizes a good trade-off between the various objectives). An attempt to reconstruct such preferences with the rule (GC$_\mu$) and an additive μ, leads to the following inequalities:

$$\mu(\{1\}) + \mu(\{3\}) + \mu(\{4\}) > \mu(\{2\}) + \mu(\{3\}) + \mu(\{4\})$$

$$\mu(\{1\}) + \mu(\{2\}) + \mu(\{3\}) > \mu(\{1\}) + \mu(\{2\}) + \mu(\{4\})$$

$$\mu(\{2\}) + \mu(\{4\}) > \mu(\{1\}) + \mu(\{3\})$$

These inequalities being contradictory, the additive rule is unable to describe the decision maker's choice.

A sound modeling of the previous example can be built using a nonadditive capacity μ, e.g., a belief function in the sense of Shafer,[33] namely, the one based on the following basic probability assignment: $m(\{1, 3\}) = 0.1$, $m(\{2, 4\}) = m(\{1, 2, 3\}) = m(\{1, 3, 4\}) = 0.2$, $m(\{1, 2, 3, 4\}) = 0.3$. Taking for μ the Shafer's belief function defined as $\mu(A) = \Sigma_{F \subseteq A}\, m(F)$, we indeed get

$$\mu(\{1, 3, 4\}) = 0.3 > \mu(\{2, 3, 4\}) = 0.2$$

$$\mu(\{1, 2, 3\}) = 0.3 > \mu(\{1, 2, 4\}) = 0.2$$

$$\mu(\{2, 4\}) = 0.2 > \mu(\{1, 3\}) = 0.1$$

Another well-known argument against additive concordance rules is that they lead to Condorcet effects, i.e., that even the strict part of global preference $\succcurlyeq$ ($\succ$) may fail to be transitive. However, there are some nonadditive concordance rules that alleviate this lack of transitivity, e.g., the one obtained when μ is a necessity or a possibility measure (see Ref. 6).

Both kinds of measures avoid the Condorcet effect.

Proposition 2. *When μ is a possibility measure (resp. a necessity measure) and the relations $\succcurlyeq_j$ on X_j, for all $j = 1, \ldots, n$ are weak orders, the preference order on N defined by Definition 2 is quasi-transitive.*

Remark that possibility and necessity measures are not the sole capacity functions that ensure the quasi-transitivity of $\succcurlyeq$ (i.e., the transitivity of $\succ$). This also is the case of some very particular probability functions (but not all, as shown in the previous example), namely, the one that encodes a lexicographic ordering of the attributes.[34]

4. A CHARACTERIZATION OF THE GC RULE

To better understand the descriptive potential of GC, we now characterize preference structures that are compatible with this rule. In this study, both X and N are supposed to be finite (consistently, each X_j admits a finite set of values). We also investigate the practical construction of the adequate instance of the rule from a given preference relation $\succcurlyeq$ on the entire multiattribute space X. This relation represents the decision maker's preferences. It is assumed to be partially observable on a sufficiently rich part of X. From this initial relation $\succcurlyeq$, one can define, for any attribute, a marginal preference relation $\succcurlyeq_j$ restricted to the jth attribute.

Definition 3. *For all $j \in N : (x_j \succcurlyeq_j y_j \Leftrightarrow$ for all $\mathbf{z} \in X$, $(\mathbf{x}\{j\}\mathbf{z}) \succcurlyeq (\mathbf{y}\{j\}\mathbf{z}))$.*

Hence, the concordance sets $C_{\succcurlyeq}(\mathbf{x}, \mathbf{y})$ are known for all $j \in N$ and all pairs $(\mathbf{x}, \mathbf{y})$:

$$C_{\succcurlyeq}(\mathbf{x}, \mathbf{y}) = \{j \in N,\ \forall\ \mathbf{z} \in X, (\mathbf{x}\{j\}\mathbf{z}) \succcurlyeq (\mathbf{y}\{j\}\mathbf{z})\}$$

Then, we introduce a first axiom, strongly enforcing the qualitative nature of the model.

Axiom NIM (Neutrality-Independence Monotony). *For all $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w} \in X$ if $[C_{\succcurlyeq}(\mathbf{x}, \mathbf{y}) \subseteq C_{\succcurlyeq}(\mathbf{z}, \mathbf{w})$ and $C_{\succcurlyeq}(\mathbf{y}, \mathbf{x}) \supseteq C_{\succcurlyeq}(\mathbf{w}, \mathbf{z})]$ then $(\mathbf{x} \succcurlyeq \mathbf{y} \Rightarrow \mathbf{z} \succcurlyeq \mathbf{w})$*

NIM says that if the set of criteria where **z** dominates **w** is larger than the set of criteria where **x** dominates **y** and the set of criteria where **w** dominates **z** is smaller than the set of criteria where **y** dominates **x**, then preferring **x** to **y** we should prefer **z** to **w** as well. NIM implies that an improvement of **x** (resp. a degradation of **y**) in relations $\succcurlyeq_j$ cannot downgrade the position of **x** in $\succcurlyeq$ (resp. improve the position of **y**). The NIM axiom is a translation to MCDM of an axiom used in social choice theory (see Ref. 29). It also can be seen as a reinforcement of the noncompensation condition used in Refs. 13 and 14 adapted to the case of weak preference relations. Under this condition, the preference of **x** over **y** only depends on one-dimensional preferences $\mathbf{x} \succcurlyeq_j \mathbf{y}$ defined by Definition 3.

We introduce a second axiom that gives to any attribute the ability of discriminating at least two elements (**x**, **y**) of X.

Axiom DI (Local Discrimination). *For all* $j \in N$, *there exists* $\mathbf{x}, \mathbf{y} \in X$, $(\mathbf{x}\{j\}\mathbf{y}) \succcurlyeq \mathbf{y}$ *and not* $(\mathbf{y} \succcurlyeq (\mathbf{x}\{j\}\mathbf{y}))$.

A reinforcement of this axiom is the one that gives to any attribute the ability of discriminating at least *three* elements (**x**, **y**, **z**) of X.

Axiom DI′. *For all* $j \in N$, *there exists* $\mathbf{x}, \mathbf{y}, \mathbf{z} \in X$

$$(\mathbf{x}\{j\}\mathbf{y}) \succcurlyeq \mathbf{y} \quad \text{and not } (\mathbf{y} \succcurlyeq (\mathbf{x}\{j\}\mathbf{y}))$$

$$(\mathbf{y}\{j\}\mathbf{z}) \succcurlyeq \mathbf{z} \quad \text{and not } (\mathbf{z} \succcurlyeq (\mathbf{y}\{j\}\mathbf{z}))$$

Finally, we consider an axiom preserving a minimal comparability between the alternatives. The relation $\succcurlyeq$ is not necessarily transitive or complete. However, the incomparability is justified for a pair (**x**, **y**) only when at least two attributes j and k are conflicting, i.e., $x_j \succ_j y_j$ and $y_k \succ_k x_k$. Such a conflict does not exist when **x** and y differ on a single attribute. This is the meaning of the following.

Axiom LC (Local Completeness). *For all* $\mathbf{x}, \mathbf{y}, \mathbf{z}, \in X$, *for all* $j \in N$, $(\mathbf{x}\{j\}\mathbf{z}) \succcurlyeq (\mathbf{y}\{j\}\mathbf{z})$ *or* $(\mathbf{y}\{j\}\mathbf{z}) \succcurlyeq (\mathbf{x}\{j\}\mathbf{z})$.

In the sequel, we consider the following five properties derived from the three fundamental axioms:

- NI (neutrality independence).[b] For all $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w} \in X$, $[C_{\succcurlyeq}(\mathbf{x}, \mathbf{y}) = C_{\succcurlyeq}(\mathbf{z}, \mathbf{w})$ and $C_{\succcurlyeq}(\mathbf{y}, \mathbf{x}) = C_{\succcurlyeq}(\mathbf{w}, \mathbf{z})] \Rightarrow (\mathbf{x} \succcurlyeq \mathbf{y} \Leftrightarrow \mathbf{z} \succcurlyeq \mathbf{w})$
- RE (reflexivity). $\succcurlyeq$ is reflexive on X
- UN (unanimity). For all $\mathbf{x}, \mathbf{y} \in X$, $C_{\succcurlyeq}(\mathbf{x}, \mathbf{y}) = N \Rightarrow \mathbf{x} \succcurlyeq \mathbf{y}$
- IND (independence). For all $A \subseteq N$; for all $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w} \in X$, $(\mathbf{x}A\mathbf{z}) \succcurlyeq (\mathbf{y}A\mathbf{z}) \Leftrightarrow (\mathbf{x}A\mathbf{w}) \succcurlyeq (\mathbf{y}A\mathbf{w})$
- CO (consistency). For all $A, B \in N$; for all $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w} \in$ X such that for all $j \in N$, ($x_j \succ_j y_j$ and $z_j \succ_j w_j$), $(\mathbf{x}A\mathbf{y}) \succcurlyeq (\mathbf{x}B\mathbf{y}) \Leftrightarrow (\mathbf{z}A\mathbf{w}) \succcurlyeq (\mathbf{z}B\mathbf{w})$
- GDI (global discrimination). There exists $\mathbf{x}, y \in X$, for all $j \in N$, $(\mathbf{x}\{j\}\mathbf{y}) \succcurlyeq \mathbf{y}$ and not$(y \succcurlyeq (\mathbf{x}\{j\}\mathbf{y}))$
- GDI′: There exists $\mathbf{x}, \mathbf{y}, \mathbf{z} \in X$, for all $j \in N$

$$(\mathbf{x}\{j\}\mathbf{y}) \succcurlyeq \mathbf{y} \quad \text{and not}(\mathbf{y} \succcurlyeq (\mathbf{x}\{j\}\mathbf{y}))$$

$$(\mathbf{y}\{j\}\mathbf{z}) \succcurlyeq \mathbf{z} \quad \text{and not}(\mathbf{z} \succcurlyeq (\mathbf{y}\{j\}\mathbf{z}))$$

Proposition 3.

(i) *NIM* $\Rightarrow$ *NI*
(ii) *LC* $\Rightarrow$ *RE*
(iii) *NIM* + *RE* $\Rightarrow$ *UN*

[b]A similar condition is introduced as a *noncompensation* axiom in Refs. 13, 14, and 35.

(iv) *NI* + *RE* ⇒ *IND*
(v) *NI* + *RE* ⇒ *CO*
(vi) *NIM* + *DI* ⇒ *GDI*
(vii) *NIM* + *DI′* ⇒ *GDI′*

The unanimity condition ensures that $\succcurlyeq$ refines the Pareto ordering of vectors. IND is the classical condition of preferential independence in multiattribute utility theory and the counterpart of the sure thing principle of DMU. It shows, among other things, that the comparison of $\mathbf{x}\{j\}\mathbf{z}$ and $\mathbf{y}\{j\}\mathbf{z}$ is independent from **z.** It makes the construction of relations $\succcurlyeq_j$ from $\succcurlyeq$ easier, using Definition 3. Indeed, thanks to IND, $\succcurlyeq_j$ can be defined simply as follows:

$$x_j \succcurlyeq_j y_j \Leftrightarrow \exists\ \mathbf{z} \in X, (\mathbf{x}\{j\}\mathbf{z}) \succcurlyeq (\mathbf{y}\{j\}\mathbf{z}) \tag{1}$$

CO means that when **x** is uniformly better than **y**, the preference between ($\mathbf{x}A\mathbf{y}$) and ($\mathbf{x}B\mathbf{y}$) only depends on the choice of A and B. This is the counterpart of the $P4$ principle of Savage, which projects the preference between acts into a likelihood relation between events. When NIM, DI, LC, and thus GDI and CO hold, it is possible to extract from the decision maker's preference $\succcurlyeq$ an importance relation on 2^N:

$$A \succcurlyeq_I B \Leftrightarrow (\exists\ \mathbf{x}, \mathbf{y} \in X : \forall\ j \in N, x_j \succ_j y_j \text{ and } (\mathbf{x}A\mathbf{y}) \succcurlyeq (\mathbf{x}B\mathbf{y})) \tag{2}$$

This definition is very natural. Indeed, when **x** is uniformly better than y, preferring $\mathbf{x}Ay$ to $\mathbf{x}By$ clearly is justified by the fact that the coalition of attributes A is considered as more important than the coalition B.

We are now in a position to establish the main result. First, let us observe that conditions NIM, DI, and LC are compatible. The additive concordance rule (Section 2.2, Example 1) indeed satisfies these conditions. The following representation theorem shows that any preference relation $\succcurlyeq$ verifying NIM, DI, and LC can be represented by a GC rule.

THEOREM 1. *If the decision maker's preference $\succcurlyeq$ satisfies NIM, DI, and LC, then there exists n complete preference relations $\succcurlyeq_1, \ldots, \succcurlyeq_n$, defined on $X_1, \ldots, X_n$ respectively, and a* monotonic and preadditive relation *$\succcurlyeq_I$ on 2^N, such that*

$$\forall\ v, \mathbf{w} \in X, v \succcurlyeq \mathbf{w} \Leftrightarrow C_{\succcurlyeq}(v, \mathbf{w}) \succcurlyeq_I C_{\succcurlyeq}(\mathbf{w}, v)$$

For any attribute j, $\succcurlyeq_j$ reveals the decision maker's preferences concerning the values of attribute X_j. The relation $\succcurlyeq_j$ can be constructed step by step, by observing the decision maker's preferences over pairs of alternatives of type ($\mathbf{x}\{j\}\mathbf{z}$), $\mathbf{y}\{j\}\mathbf{z}$) for an arbitrary **z.** This observation is even simpler if relations $\succcurlyeq_j$ are supposed to be transitive or quasi-transitive ($\succ_j$ transitive). Equation 2 also provides a constructive method to derive the importance relation $\succcurlyeq_I$ and thus completes the construction of the model. Notice that the entire construction is based on pairwise comparisons. Such comparisons do not require a prohibitive cognitive effort because they only concern alternatives having simple profiles.

Moreover, such comparisons do not require any explicit questioning but can be inferred by observing real choices performed by the decision maker.

The use of a preadditive relation of importance is not necessary in a GC rule in the sense that Theorem 1 only proves the *existence* of a preadditive $\succcurlyeq_I$ involved in a GC rule representing $\succcurlyeq$. If we start from a more general monotonic $\succcurlyeq_M$ (possibly not preadditive) and construct $\succcurlyeq$ via the GC rule, we still get a preference satisfying NIM, DI, and LC,[c] which thus induces a preadditive $\succcurlyeq_I$ by Equation 2.

The reason for this apparent paradox is the following. First note that $C_{\succcurlyeq}(\mathbf{x}, \mathbf{y}) \cup C_{\succcurlyeq}(\mathbf{y}, \mathbf{x}) = N$. So only the pairs (A, B) such that $A \cup B = N$ are compared in the GC rule. The following propositions can be proved.

Proposition 4. $A \succcurlyeq_I B \Leftrightarrow A \cup \bar{B} \succcurlyeq_M B \cup \bar{A}$

Proposition 5. $A \succcurlyeq_I B \Leftrightarrow A \succcurlyeq_M B$ *whenever* $A \cup B = N$

So, $\succcurlyeq_I$ and $\succcurlyeq_M$ coincide on the useful part of $2^N \times 2^N$ and the nonpreadditivity of $\succcurlyeq_M$ cannot be revealed by observing a decision maker that would use a GC rule. Example 2 is a typical case where a decision maker will use a nonpreadditive relation $\succcurlyeq_M$ induced by a belief function to describe criteria importance. In this example, criteria importance also could be represented by any preadditive relation $\succcurlyeq'_M$ such that $A \succcurlyeq_M B$ and $A \succcurlyeq'_M B$ coincide for $A \cup B = N$. The impossibility of using an additive capacity in this case stresses the gap existing between preadditivity and additivity.

Let us focus in more detail on the relation $\succcurlyeq_I$ between the coalitions of criteria that we induced by the representation theorem (Theorem 1), namely, on the restriction of its strict part $\succ_I$ to disjoint events. It is easy to show Proposition 6.

Proposition 6. *If the decision maker's preference* $\succcurlyeq$ *satisfies NIM, DI, and LC, then* $\succ_I$, *the strict part of the relation* $\succcurlyeq_I$ *defined by Equation 2 satisfies*

(i) $N \succ_I \emptyset$
(ii) $\succ_I$ *is irreflexive*
(iii) $\succ_I$ *is monotonic* ($A \succ_I B \Rightarrow A \cup C \succ_I B$ *and* $A \succ_I B \cup C \Rightarrow A \succ_I B$)
(iv) When $\succcurlyeq$ *is quasi-transitive and DI′ holds, then* $\succ_I$ *is transitive and for all* A, B, C *disjoint sets* $A \cup B \succ_I C$ *and* $A \cup C \succ_I B \Rightarrow A \succ_I B \cup C$
(v) For all $A \neq \emptyset$, $A \succ_I \emptyset$

Note that Proposition 6(iv) applies as soon as the preference scales pertaining to criteria contain more than two levels (i.e., they are not Boolean) and the decision maker is rational in his/her decisions, i.e., does not exhibit an inconsistent global preference relation where strict preference would not be transitive. The obtained property on $\succ_I$ was first suggested by Dubois and Prade[36] as well as Friedman and

[c]It can be shown that any preference relation $\succcurlyeq$ constructed from an importance relation $\succcurlyeq_M$ using a GC rule satisfies NIM, DI, and LC as soon as the four following properties hold: for all $j \in N$, $\succcurlyeq_j$ is complete, for all $j \in N$, there exist $(\mathbf{x}, \mathbf{y}) \in X^2 x_j \succ_j y_j$, $\succcurlyeq_M$ is reflexive, monotonic, and for all $j \in N$, $N \succ_M N \backslash \{j\}$.

Halpern[37] in the setting of nonmonotonic reasoning. An obvious consequence of Proposition 6(iv) is that if $A >_I B$ and $A >_I C$, then $A >_I B \cup C$ for disjoint coalitions of criteria; $A >_I B$ means that the importance of coalition B is negligible in front of A. Relation $>_I$ is also called an acceptance relation in the field of likelihood representation.[38] It can be shown that such a partial order can be represented by families of possibility relations.[36] Thus, we get the following result.

Corollary 1. *If the decision maker's preference $\succcurlyeq$ is complete, quasi-transitive, and satisfies NIM, DI′, and LC then there exists n complete preference relations $\succcurlyeq_1, \ldots, \succcurlyeq_n$, defined on $X_1, \ldots, X_n$, respectively, and there exists a family $\mathcal{F}$ of nondogmatic[d] possibility distributions on N such that*

$$\forall\, \mathbf{x}, \mathbf{y} \in X, \mathbf{x} \succ \mathbf{y} \Leftrightarrow \forall\, \pi_i \in \mathcal{F}, \Pi_i(C_{\succ}(\mathbf{x}, \mathbf{y})) > \Pi_i(C_{\succ}(\mathbf{y}, \mathbf{x}))$$

$$\Leftrightarrow \forall\, \pi_i \in \mathcal{F}, \exists\, j \in C_{\succ}(\mathbf{x}, \mathbf{y}) \text{ such that } \pi_i(j) > \pi_i(j')\forall\, j' \in C_{\succ}(y, \mathbf{x})$$

$$\Leftrightarrow \forall\, \pi_i \in \mathcal{F}, Nec_i(C_{\succcurlyeq}(\mathbf{x}, \mathbf{y})) > Nec_i(C_{\succcurlyeq}(\mathbf{y}, \mathbf{x})),$$

$$\forall\, \mathbf{x}, \mathbf{y} \in X, \mathbf{x} \succcurlyeq \mathbf{y} \Leftrightarrow \exists\, \pi_i \in \mathcal{F}, \Pi_i(C_{\succ}(\mathbf{x}, \mathbf{y})) \geqslant \Pi_i(C_{\succ}(\mathbf{y}, \mathbf{x}))$$

$$\Leftrightarrow \exists\, \pi_i \in \mathcal{F}, Nec_i(C_{\succcurlyeq}(\mathbf{x}, \mathbf{y})) \geqslant Nec_i(C_{\succcurlyeq}(\mathbf{y}, \mathbf{x}))$$

Conversely, any concordance rule based on a family of nondogmatic possibility distributions and a set of marginal utility rankings satisfies NIM and LC and is quasi-transitive, provided that the marginal utility rankings are weak orders.

Proposition 7. *Let $\mathcal{F}$ be a family of nondogmatic possibility distributions on N and $\{\succcurlyeq_1, \ldots, \succcurlyeq_n\}$ be a set of weak orders defined on $\{X_1, \ldots, X_n\}$, respectively. The preference order $\succcurlyeq$ defined by*

$$\forall\, \mathbf{x}, \mathbf{y} \in X, \mathbf{x} \succ \mathbf{y} \Leftrightarrow \forall\, \pi_i \in \mathcal{F}, Nec_i(C_{\succcurlyeq}(\mathbf{x}, \mathbf{y})) > Nec_i(C_{\succcurlyeq}(\mathbf{y}, \mathbf{x}))$$

$$\forall\, \mathbf{x}, \mathbf{y} \in X, \mathbf{x} \succcurlyeq \mathbf{y} \Leftrightarrow not\ (\mathbf{y} \succ \mathbf{x})$$

satisfies NIM and LC and is quasi-transitive. It satisfies DI (resp. DI′) iff each of the marginal utility rankings $\succcurlyeq_i$ is such that there exists $x_i, y_i \in X_i, x_i \succ_i y_i$ (resp. there exists $x_i, y_i, z_i \in X_i, x_i \succ_i y_i$ and $y_i \succ_i z_i$).

This shows the high compatibility of rational concordance rules with possibility theory; the previous theorem indeed means that the global relation can be viewed as the merging of a family of necessity-based concordance rules of the form

$$\mathbf{x} \succ_{\mathrm{Nec}_i} \mathbf{y} \Leftrightarrow \mathrm{Nec}_i(C_{\succcurlyeq}(\mathbf{x}, \mathbf{y})) > \mathrm{Nec}_i(C_{\succcurlyeq}(\mathbf{y}, \mathbf{x}))$$

$$\text{i.e.,}\quad \mathbf{x} \succcurlyeq_{\mathrm{Nec}_i} \mathbf{y} \Leftrightarrow \mathrm{Nec}_i(C_{\succcurlyeq}(\mathbf{x}, \mathbf{y})) \geqslant \mathrm{Nec}_i(C_{\succcurlyeq}(\mathbf{y}, \mathbf{x}))$$

When there is only one necessity measure in the family, we recover the necessity-based concordance rule.

[d] A possibility distribution π is said to be nondogmatic iff $\pi(j) > 0$, for all $j = 1, \ldots, n$.

At this point, one should wonder whether the concordance rules based on the dual of a necessity relation, i.e., the possibility-based concordance rules, are as attractive as those based on necessity relations. The answer is actually negative. Indeed, the concordance rule based on a possibility measure turns out to be highly drastic. This feature appears even if some graduality of importance is available through the possibility degrees associated with the criteria; the rule only considers the criteria that receive a possibility degree of 1, and neglects the others, even if they have a positive importance.

PROPOSITION 8. *If $\succcurlyeq$ is defined according to Definition 2 from a set of complete relations $\succcurlyeq_j$ on X_j, for all $j = 1, \ldots, n$ and a possibility measure Π on 2^N, then*

$$\forall\, \mathbf{x}, \mathbf{y} \in X, (\mathbf{x} \succcurlyeq \mathbf{y} \Leftrightarrow [\exists\, i \in N \text{ such that } \Pi(\{i\}) = 1 \text{ and } x_i \succcurlyeq_i y_i])$$

Note that unanimity on the criteria of possibility degree 1 is not required for ensuring the global preference. It is enough that one of the most important criteria prefers **x** to **y** in the wide sense, for **x** to be preferred to **y**. None of the other criteria can prevent it.

Such bad-behaved rules actually are ruled out by our axiomatics in the set of GC rules. Indeed, they do not satisfy axiom DI. Any possibility-based concordance rule relying on a distribution π that admits more than one state of possibility 1 forms a counterexample.

PROPOSITION 9. *If $\succcurlyeq$ is defined according to Definition 2 from a set of complete relations $\succcurlyeq_j$ on X_j, for all $j = 1, \ldots, n$ and a possibility measure Π, then*

$$\succcurlyeq \text{ satisfies DI} \Leftrightarrow \exists!\, i \in N \text{ such that } \Pi(\{i\}) = 1$$

So, Theorem 1 accounts for necessity-based concordance rules (among others) but excludes possibility-based ones. Finally, it should be noted that GC rules are not the ultimate answer to qualitative MCDM problems. The following counterexample shows that, despite its apparent generality, GC rules are not always able to represent ordinal preferences on a multiattribute space.

Example 3. Let us consider the following example mentioned in Ref. 27, giving the grades obtained by four students **a**, **b**, **c**, and **d**, according to three courses: Physics, Math, and Economics.

	Physics	*Maths*	*Economics*
a	18	12	6
b	18	7	11
c	5	17	8
d	5	12	13

On the basis of these evaluations it is felt that Student **a** should be ranked before Student **b**. Although Student **a** has a low grade in Economics, he has reasonably good grades in both Math and Physics, which makes a good candidate for an Engineering program; Student **b** is weak in Math and it seems difficult to recommend him for Engineering or Economics programs. Student **c** has two low grades and it seems difficult to recommend him at all. Student **d** is preferred to Student **c** because he obtained reasonable grades in Math and Economics and can be recommended for the program in Economics. The question is, can we represent preferences $\mathbf{a} \succ \mathbf{b}$ and $\mathbf{d} \succ \mathbf{c}$ using a GC rule? In this particular case, the answer is negative. Indeed we have

$$C_{\succ}(\mathbf{a}, \mathbf{b}) = \{1, 2\} \quad C_{\succcurlyeq}(\mathbf{b}, \mathbf{a}) = \{1, 3\}$$

$$C_{\succcurlyeq}(\mathbf{d}, \mathbf{c}) = \{1, 3\} \quad C_{\succcurlyeq}(\mathbf{c}, \mathbf{d}) = \{1, 2\}$$

Trying to represent such preferences using a GC rule $\mathbf{a} \succ \mathbf{b}$ *implies* $C_{\succcurlyeq}(\mathbf{a}, \mathbf{b}) \succ_I C_{\succcurlyeq}(\mathbf{b}, \mathbf{a})$ and $\mathbf{d} \succ \mathbf{c}$ implies $C_{\succcurlyeq}(\mathbf{d}, \mathbf{c}) \succ_I C_{\succcurlyeq}(\mathbf{c}, \mathbf{d})$. Observing that $C_{\succcurlyeq}(\mathbf{a}, \mathbf{b}) = C_{\succcurlyeq}(\mathbf{c}, \mathbf{d})$ and $C_{\succcurlyeq}(\mathbf{b}, \mathbf{a}) = C_{\succcurlyeq}(\mathbf{d}, \mathbf{c})$, the two previous inequalities are contradictory. Note that, as shown in Ref. 37, a comparison model based on a weighted sum of grades is not able to represent such preferences either.

5. CONCLUSIONS

In this study we have proposed a generalization of MCDM's additive concordance rules. An example is given where the classical concordance rule is unable to describe the decision maker's choice, although a CA rule based on a nonadditive capacity (namely, a belief function) could solve the example, thus showing the interest of GC rules. Our general rule also encompasses the necessity-based concordance rule proposed by Ref. 8 for qualitative DMU. It is worth noticing that this framework is not restricted to capacity functions but can be applied whenever the relative importance of the coalitions of criteria can be encoded by a reflexive and monotonic (partial) relation. Then, we have proposed a well-founded axiomatic framework for those multiattribute decision problems where the scales pertaining to the different attributes and the one describing the importance of the coalitions are not commensurate. Taking it a step further, a parallel between MCDM and DMU allowed us to import a representation theorem that states that the satisfaction of a few simple additional principles (namely, completeness and strong discrimination) implies a highly possibilistic behavior of the concordance rule. In this case, we indeed get a semantics in terms of necessity relations, i.e., in terms of hierarchies of oligarchies.

It is clear that GC rules are not the ultimate answer to qualitative MCDM problems. One can easily imagine situations in which the preference of a decision maker cannot be expressed in this way, e.g., when the preference between **x** and y depends on a third alternative z, as in MCDM filtering methods[39] or in counterparts to stochastic dominance in DMU.[7] Last, as suggested by Example 2, the approach presented here could benefit multiagent fusion problems[17–19] once adapted to a logical setting.

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APPENDIX A

Proof of Proposition 1. Notice that, for all $j \in N$, $u_j(\mathbf{x}^* A \mathbf{x}_*) = 1$ if $j \in A$, $u_j(\mathbf{x}^* A \mathbf{x}_*) = 0$ otherwise. Because ψ_μ is coincident, we have $\psi_\mu(u_1(\mathbf{x}^* A \mathbf{x}_*), \ldots, u_n(\mathbf{x}^* A \mathbf{x}_*)) = \mu(A)$. Moreover, ψ_μ being idempotent we have $\psi_\mu(u_1(a), \ldots, u_n(a)) = \alpha$. Hence, we get $\psi_\mu(u_1(\mathbf{x}^* A \mathbf{x}_*), \ldots, u_n(\mathbf{x}^* A \mathbf{x}_*)) \geqslant \psi_\mu(u_1(\mathbf{a}), \ldots, u_n(\mathbf{a})) \Leftrightarrow \mu(A) \geqslant \alpha$. Because the left side of this equation is equivalent to $(\mathbf{x}^* A \mathbf{x}_*) \succcurlyeq_\mu \mathbf{a}$, we get the result.

Proof of Proposition 2. Let us consider three alternatives $\mathbf{x}$, y, and z. Because the relations $\succcurlyeq_j$ on X_i are weak orders, for all $i = 1, \ldots, n$, N can be partitioned as follows:

$$A = \{i, \mathbf{x} \succ_i \mathbf{y} \succ_i \mathbf{z}\} \quad B = \{i, \mathbf{x} \succ_i \mathbf{y} \sim_i \mathbf{z}\} \quad C = \{i, \mathbf{x} \succ_i \mathbf{z} \succ_i \mathbf{y}\}$$

$$D = \{i, \mathbf{y} \succ_i \mathbf{x} \succ_i \mathbf{z}\} \quad E = \{i, \mathbf{y} \succ_i \mathbf{x} \sim_i \mathbf{z}\} \quad F = \{i, \mathbf{y} \succ_i \mathbf{z} \succ_i \mathbf{x}\}$$

$G = \{i, \mathbf{z} \succ_i \mathbf{x} \succ_i \mathbf{y}\}$ $H = \{i, \mathbf{z} \succ_i \mathbf{x} \sim_i \mathbf{y}\}$ $I = \{i, \mathbf{z} \succ_i \mathbf{y} \succ_i \mathbf{x}\}$

$J = \{i, \mathbf{x} \sim_i \mathbf{y} \succ_i \mathbf{z}\}$ $K = \{i, \mathbf{x} \sim_i \mathbf{z} \succ_i \mathbf{x}\}$ $L = \{i, \mathbf{y} \sim_i \mathbf{z} \succ_i \mathbf{x}\}$

$M = \{i, \mathbf{x} \sim_i \mathbf{y} \sim_i \mathbf{z}\}$

When Definition 2 is used, we obviously get a complete relation $\succcurlyeq$. Suppose that $\mathbf{x} \succ \mathbf{y}$, $\mathbf{y} \succ \mathbf{z}$, and that not $\mathbf{x} \succ \mathbf{z}$, i.e., $\mathbf{z} \succcurlyeq \mathbf{x}$.

When μ is a necessity relation, this writes

- $\mathbf{x} \succ \mathbf{y}$: $\text{Nec}(C_{\succcurlyeq}(\mathbf{x}, \mathbf{y})) > \text{Nec}(C_{\succcurlyeq}(\mathbf{y}, \mathbf{x}))$, i.e., $\Pi(C_{\succ}(\mathbf{x}, \mathbf{y})) > \Pi(C_{\succ}(\mathbf{y}, \mathbf{x}))$, i.e., $\max(\Pi(A), \Pi(B), \Pi(C), \Pi(G), \Pi(K)) > \max(\Pi(D), \Pi(E), \Pi(F), \Pi(I), \Pi(L))$
- $\mathbf{y} \succ \mathbf{z}$: $\text{Nec}(C_{\succcurlyeq}(\mathbf{y}, \mathbf{z})) > \text{Nec}(C_{\succcurlyeq}(\mathbf{z}, \mathbf{y}))$, i.e., $\Pi(C_{\succ}(\mathbf{y}, \mathbf{z})) > \Pi(C_{\succ}(\mathbf{y}, \mathbf{z}))$, i.e., $\max(\Pi(A), \Pi(D), \Pi(E), \Pi(F), \Pi(J)) > \max(\Pi(C), \Pi(G), \Pi(H), \Pi(I), \Pi(K))$
- $\mathbf{z} \succcurlyeq \mathbf{x}$: $\text{Nec}(C_{\succcurlyeq}(\mathbf{z}, \mathbf{x})) \geqslant \text{Nec}(C_{\succcurlyeq}(\mathbf{x}, \mathbf{z}))$, i.e., $\Pi(C_{\succ}(\mathbf{z}, \mathbf{x})) \geqslant \Pi(C_{\succ}(\mathbf{x}, \mathbf{z}))$, i.e., $\max(\Pi(F), \Pi(G), \Pi(H), \Pi(I), \Pi(L)) \geqslant \max(\Pi(A), \Pi(B), \Pi(C), \Pi(D), \Pi(J))$

Thus, we get a system of equations of the form

$$\begin{cases} \max(a, b, c, g, k) > \max(d, e, f, i, l) \\ \max(a, d, e, f, j) > \max(c, g, h, i, k) \\ \max(f, g, h, i, l) \geqslant \max(a, b, c, d, j) \end{cases}$$

which is inconsistent.

When μ is a possibility relation, $\mathbf{x} \succ \mathbf{y}$, $\mathbf{y} \succ z$, and $\mathbf{z} \succcurlyeq \mathbf{x}$. writes

- $\mathbf{x} \succ \mathbf{y}$: $\Pi(C_{\succcurlyeq}(\mathbf{x}, \mathbf{y})) > \Pi(C_{\succcurlyeq}(\mathbf{y}, \mathbf{x}))$, i.e., $\max(\Pi(A), \Pi(B), \Pi(C), \Pi(G), \Pi(K), \Pi(H), \Pi(J), \Pi(M)) > \max(\Pi(D), \Pi(E), \Pi(F), \Pi(I), \Pi(L), \Pi(H), \Pi(J), \Pi(M))$
- $\mathbf{y} \succ \mathbf{z}$: $\Pi(C_{\succcurlyeq}(\mathbf{y}, \mathbf{z})) > \Pi(C_{\succcurlyeq}(\mathbf{z}, \mathbf{y}))$, i.e., $\max(\Pi(A), \Pi(D), \Pi(E), \Pi(F), \Pi(J), \Pi(B), \Pi(L), \Pi(M)) > \max(\Pi(C), \Pi(G), \Pi(H), \Pi(I), \Pi(K), \Pi(B), \Pi(L), \Pi(M))$
- $\mathbf{z} \succcurlyeq \mathbf{x}$: $\Pi(C_{\succcurlyeq}(\mathbf{z}, \mathbf{x})) \geqslant \Pi(C_{\succcurlyeq}(\mathbf{x}, \mathbf{z}))$, i.e., $\max(\Pi(F), \Pi(G), \Pi(H), \Pi(I), \Pi(L), \Pi(E), \Pi(K), \Pi(M)) \geqslant \max(\Pi(A), \Pi(B), \Pi(C), \Pi(D), \Pi(J), \Pi(E), \Pi(\mathrm{K}), \Pi(\mathrm{M}))$

Again, we get a system of equations which is inconsistent:

$$\begin{cases} \max(a, b, c, g, k, h, j, m) > \max(d, e, f, i, l, h, j, m) \\ \max(a, d, e, f, j, b, l, m) > \max(c, g, h, i, k, b, l, m) \\ \max(f, g, h, i, l, e, k, m) \geqslant \max(a, b, c, d, j, e, k, m) \end{cases}$$

Proof of Proposition 3(i). From $C_{\succcurlyeq}(\mathbf{x}, \mathbf{y}) = C_{\succcurlyeq}(\mathbf{z}, \mathbf{w})$ and $C_{\succcurlyeq}(\mathbf{y}, \mathbf{x}) = C_{\succcurlyeq}(\mathbf{w}, \mathbf{z})$, we can apply NIM twice, one time for $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, and $\mathbf{w}$, and we get $\mathbf{x} \succcurlyeq \mathbf{y} \Rightarrow \mathbf{z} \succcurlyeq \mathbf{w}$, and one time for $\mathbf{z}$, $\mathbf{w}$, $\mathbf{x}$, and $\mathbf{y}$, and we get $\mathbf{z} \succcurlyeq \mathbf{w} \Rightarrow \mathbf{x} \succcurlyeq \mathbf{y}$. Thus, when NIM holds, ($C_{\succcurlyeq}(\mathbf{x}, \mathbf{y}) = C_{\succcurlyeq}(\mathbf{z}, \mathbf{w})$ and $C_{\succcurlyeq}(\mathbf{y}, \mathbf{x}) = C_{\succcurlyeq}(\mathbf{w}, \mathbf{z})$) implies that $\mathbf{x} \succcurlyeq \mathbf{y} \Leftrightarrow \mathbf{z} \succcurlyeq \mathbf{w}$, i.e., NI holds.

Proof of Proposition 3(ii). RE is retrieved when applying LC with $\mathbf{x} = \mathbf{y} = \mathbf{z}$. In this case, it indeed writes; for all $\mathbf{x} \in X$, for all $j \in N$, $(\mathbf{x}\{j\}\mathbf{x}) \succcurlyeq (\mathbf{x}\{j\}\mathbf{x})$, or $(\mathbf{x}\{j\}\mathbf{x}) \succcurlyeq (\mathbf{x}\{j\}\mathbf{x})$, i.e., for all $\mathbf{x} \in X$, $\mathbf{x} \succcurlyeq \mathbf{x}$.

Proof of Proposition 3(iii). By Definition 3, for all $\mathbf{x} \in X$, $x_j \succsim_j x_j \Leftrightarrow \mathbf{x}\{j\}\mathbf{z} \succsim \mathbf{x}\{j\}\mathbf{z}$, for all $\mathbf{z} \in X$. Because $\mathbf{x}\{j\}\mathbf{z} \succsim \mathbf{x}\{j\}\mathbf{z}$ by RE, we deduce that relations $\succsim_j$, $j = 1, \dots, n$ are reflexive. Thus, whatever $\mathbf{x}$, $C_{\succsim}(\mathbf{x}, \mathbf{x}) = N$.

Consider now two vectors $\mathbf{x}$ and $\mathbf{y}$ such that $C_{\succsim}(\mathbf{x}, \mathbf{y}) = N$. It holds that $C_{\succsim}(\mathbf{x}, \mathbf{x}) = N \subseteq C_{\succsim}(\mathbf{x}, \mathbf{y})$ and that $C_{\succsim}(\mathbf{x}, \mathbf{x}) = N \supseteq C_{\succsim}(\mathbf{y}, \mathbf{x})$. So, applying NIM to the quadruplets $(\mathbf{x}, \mathbf{x}, \mathbf{x}, \mathbf{y})$, we get $\mathbf{x} \succsim \mathbf{x} \Rightarrow \mathbf{x} \succsim \mathbf{y}$. Because $\mathbf{x} \succsim \mathbf{x}$ holds by RE, we get $\mathbf{x} \succsim \mathbf{y}$, which proves UN.

Proof of Proposition 3(iv). For any $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w} \in X$, for any $A \subseteq N$, we have: $C_{\succsim}(\mathbf{x}A\mathbf{z}, yA\mathbf{z}) = C_{\succsim}(\mathbf{x}A\mathbf{w}, \mathbf{y}A\mathbf{w})$ because of the reflexivity of the $\succsim_j$, $j = 1, \dots, n$, that follows from reflexivity (see the previous proof). In the same way, $C_{\succsim}(\mathbf{y}A\mathbf{z}, \mathbf{x}A\mathbf{z}) = C_{\succsim}(\mathbf{y}A\mathbf{w}, \mathbf{x}A\mathbf{w})$. Thus, we get by NI $(\mathbf{x}A\mathbf{x} \succsim \mathbf{y}A\mathbf{x} \Leftrightarrow \mathbf{x}A\mathbf{w} \succsim \mathbf{y}A\mathbf{w})$.

Proof of Proposition 3(v). Let us consider two coalitions $A, B \subseteq N$ and four vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w} \in X$ such that for all $j \in N$, $(x_j \succ_j y_j$ and $z_j \succ_j w_j)$

- For any $j \in A \cap B$, $(\mathbf{x}A\mathbf{y})_j = (\mathbf{x}B\mathbf{y})_j = x_j$ and $(\mathbf{z}A\mathbf{w})_j = (\mathbf{z}B\mathbf{w})_j = z_j$. Thus, by RE of the $\succsim_j$ (that follows from RE) $(\mathbf{x}A\mathbf{y})_j \sim_j (\mathbf{x}B\mathbf{y})_j$ and $(\mathbf{z}A\mathbf{w})_j \sim_j (\mathbf{z}B\mathbf{w})_j$
- For any $j \in A \cap \bar{B}$, $(\mathbf{x}A\mathbf{y})_j = x_j$, $(\mathbf{x}B\mathbf{y})_j = y_j$ and $(\mathbf{z}A\mathbf{w})_j = z_j$, $(\mathbf{z}B\mathbf{w})_j = w_j$; thus, $(\mathbf{x}A\mathbf{y})_j \succ_j (\mathbf{x}B\mathbf{y})_j$ and $(\mathbf{z}A\mathbf{w})_j \succ_j (\mathbf{z}B\mathbf{w})_j$.
- For any $j \in \bar{A} \cap B$, $(\mathbf{x}A\mathbf{y})_j = y_j$, $(\mathbf{x}B\mathbf{y})_j = x_j$ and $(\mathbf{z}A\mathbf{w})_j = w_j$, $(\mathbf{z}B\mathbf{w})_j = z_j$; thus, $(\mathbf{x}B\mathbf{y})_j \succ_j (\mathbf{x}A\mathbf{y})_j$ and $(\mathbf{z}B\mathbf{w})_j \succ_j (\mathbf{z}A\mathbf{w})_j$.
- For any $j \in \bar{A} \cap \bar{B}$, $(\mathbf{x}A\mathbf{y})_j = (\mathbf{x}B\mathbf{y})_j = y_j$ and $(\mathbf{z}A\mathbf{w})_j = (\mathbf{z}B\mathbf{w})_j = w_j$; thus, $(\mathbf{x}A\mathbf{y})_j \sim_j (\mathbf{x}B\mathbf{y})_j$ and $(\mathbf{z}A\mathbf{w})_j \sim_j (\mathbf{z}B\mathbf{w})_j$ because of the RE of the S_j.

Thus, we have: $C_{\succ}(\mathbf{x}A\mathbf{y}, \mathbf{x}B\mathbf{y}) = C_{\succ}(\mathbf{z}A\mathbf{w}, \mathbf{z}B\mathbf{w})$, $C_{\succ}(\mathbf{x}B\mathbf{y}, \mathbf{x}A\mathbf{y}) = C_{\succ}(\mathbf{z}B\mathbf{w}, \mathbf{z}A\mathbf{w})$ and $C_{\sim}(\mathbf{x}A\mathbf{y}, \mathbf{x}B\mathbf{y}) = C_{\sim}(\mathbf{z}A\mathbf{w}, \mathbf{z}B\mathbf{w})$. Thus, $C_{\succsim}(\mathbf{x}A\mathbf{y}, \mathbf{x}B\mathbf{y}) = C_{\succsim}(\mathbf{z}A\mathbf{w}, \mathbf{z}B\mathbf{w})$ and $C_{\succsim}(\mathbf{x}B\mathbf{y}, \mathbf{x}A\mathbf{y}) = C_{\succsim}(\mathbf{z}B\mathbf{w}, \mathbf{z}A\mathbf{w})$. By NI, this implies $(\mathbf{x}A\mathbf{y}) \succsim (\mathbf{x}B\mathbf{y}) \Leftrightarrow (\mathbf{z}A\mathbf{w}) \succsim (\mathbf{z}B\mathbf{w})$.

Proof of Propositions 3(vi) and (vii). By DI, we know that for all $j \in N$, there exist two vectors $\mathbf{a}^j$ and $\mathbf{b}^j$ such that $(\mathbf{a}^j\{j\}\mathbf{b}^j) \succ \mathbf{b}^j$. Thus, by NIM, we get for any $\mathbf{c}$, $\mathbf{a}^j\{j\}\mathbf{c} \succ \mathbf{b}^j\{j\}\mathbf{c}$. So, it holds that for all $j \in N$, $\mathbf{a}^j_j \succ_j \mathbf{b}^j_j$. Hence, the vectors $\mathbf{x} = (a^1_1, \dots, a^n_n)$ and $\mathbf{y} = (b^1_1, \dots, b^n_n)$ are such that $x_j \succ_j y_j$, $j = 1, \dots, n$. So, GDI holds. The proof of item (vii) is similar.

Proof of Theorem 1.

Proof of the Completeness of $\succsim_j$. The proof is direct from LC. Indeed LC means that for all $\mathbf{x}, \mathbf{y} \in X$ and whatever $j \in N$, it holds that: for all $\mathbf{z} \in X$, $(\mathbf{x}\{j\}\mathbf{z}) \succsim (\mathbf{y}\{j\}\mathbf{z})$ or for all $\mathbf{z} \in X$, $(\mathbf{y}\{j\}\mathbf{z}) \succsim (\mathbf{x}\{j\}\mathbf{z})$. By Definition 3, this means that $x_j \succsim_j y_j$ or $y_j \succsim_j x_j$. Because this holds for any pair $(\mathbf{x}, \mathbf{y})$, this proves that the relations $\succsim_j$, $j = 1, \dots, n$ are complete.

Proof of the main result. Let us now consider a pair $(\boldsymbol{v}, \mathbf{w}) \in X^2$ such as $\boldsymbol{v} \succsim \mathbf{w}$. Let us denote $A = C_{\succsim}(\boldsymbol{v}, \mathbf{w})$ and $B = C_{\succsim}(\mathbf{w}, \boldsymbol{v})$.

Consider the relation $\succcurlyeq_I$ between coalitions defined by Equation 2. Because NIM, DI, and LC are assumed, we know that CO and GDI hold (Proposition 3) and that there exists $\mathbf{x}$, y such that $A \succcurlyeq_I B$ iff $(\mathbf{x}A\mathbf{y}) \succcurlyeq (\mathbf{x}B\mathbf{y})$.

Now, for any $j \in N$, $\succcurlyeq_j$ is complete (by LC). So, for criteria j, only three cases are possible:

$v_j \succ_j w_j$: $(\mathbf{x}A\mathbf{y})_j = x_j$ and $(\mathbf{x}B\mathbf{y})_j = y_j$; thus, $(\mathbf{x}A\mathbf{y})_j \succ_j (\mathbf{x}B\mathbf{y})_j$
$w_j \succ_j v_j$: $(\mathbf{x}A\mathbf{y})_j = y_j$ and $(\mathbf{x}B\mathbf{y})_j = x_j$; thus, $(\mathbf{x}B\mathbf{y})_j \succ_j (\mathbf{x}A\mathbf{y})_j$
$v_j \sim_j w_j$: $(\mathbf{x}A\mathbf{y})_j = x_j$ and $(\mathbf{x}B\mathbf{y})_j = x_j$; thus, $(\mathbf{x}A\mathbf{y})_j \sim_j (\mathbf{x}B\mathbf{y})_j$

So, $C_{\succ}(\boldsymbol{v}, \mathbf{w}) = C_{\succ}(\mathbf{x}A\mathbf{y}, \mathbf{x}B\mathbf{y})$, $C_{\succ}(\mathbf{w}, \boldsymbol{v}) = C_{\succ}(\mathbf{x}B\mathbf{y}, \mathbf{x}A\mathbf{y})$, $C_{\sim}(\boldsymbol{v}, \mathbf{w}) = C_{\sim}(\mathbf{x}A\mathbf{y}, \mathbf{x}B\mathbf{y})$. Thus, $C_{\succcurlyeq}(\boldsymbol{v}, \mathbf{w}) = C_{\succcurlyeq}(\mathbf{x}A\mathbf{y}, \mathbf{x}B\mathbf{y})$ and $C_{\succcurlyeq}(\mathbf{w}, \boldsymbol{v}) = C_{\succcurlyeq}(\mathbf{x}B\mathbf{y}, \mathbf{x}A\mathbf{y})$. By NI, this proves that $v \succcurlyeq w \Leftrightarrow (\mathbf{x}Ay) \succcurlyeq (\mathbf{x}By)$. Moreover, we know that $(\mathbf{x}A\mathbf{y}) \succcurlyeq (\mathbf{x}B\mathbf{y}) \Leftrightarrow A \succcurlyeq_I B$. Thus, $\boldsymbol{v} \succcurlyeq \mathbf{w} \Leftrightarrow C_{\succcurlyeq}(\boldsymbol{v}, \mathbf{w}) \succcurlyeq_I C_{\succcurlyeq}(\mathbf{w}, \boldsymbol{v})$.

Proof of the Monotony of $\succcurlyeq_I$. Let us now show that $\succcurlyeq_I$ is monotonic. Consider two coalitions of attributes A and B such that $A \succcurlyeq_I B$. By definition, this means that there are two vectors $\mathbf{x}$ and $\mathbf{y}$ such that $x_j \succ_j y_j$, for all $j \in N$, and such that $(\mathbf{x}A\mathbf{y}) \succcurlyeq (\mathbf{x}B\mathbf{y})$. Note that $C_{\succcurlyeq}(\mathbf{x}A\mathbf{y}, \mathbf{x}B\mathbf{y}) = A \cup \bar{B}$ [this actually holds for (A, B) being any pair of events]. Then, for any other coalition $C \subseteq N$ such that $A \cap C = \varnothing$: $C_{\succcurlyeq}(\mathbf{x}A\mathbf{y}, \mathbf{x}B\mathbf{y}) \subseteq C_{\succcurlyeq}(\mathbf{x}(A \cup C)\mathbf{y}, \mathbf{x}B\mathbf{y})$, $C_{\succcurlyeq}(\mathbf{x}B\mathbf{y}, \mathbf{x}A\mathbf{y}) \supseteq C_{\succcurlyeq}(\mathbf{x}B\mathbf{y}, \mathbf{x}(A \cup C)\mathbf{y})$, and $(\mathbf{x}A\mathbf{y}) \succcurlyeq (\mathbf{x}B\mathbf{y})$ imply $(\mathbf{x}(A \cup C)\mathbf{y} \succcurlyeq (\mathbf{x}B\mathbf{y})$ via NIM. Thus, $A \cup C \succcurlyeq_I B$. Similarly, if $A \succcurlyeq_I B \cup C$, and then $C_{\succcurlyeq}(\mathbf{x}Ay, \mathbf{x}(B \cup C)\mathbf{y}) \subseteq C_{\succcurlyeq}(\mathbf{x}A\mathbf{y}, \mathbf{x}B\mathbf{y})$, $C_{\succcurlyeq}(\mathbf{x}(B \cup C)\mathbf{y}, \mathbf{x}A\mathbf{y}) \supseteq C_{\succcurlyeq}(\mathbf{x}B\mathbf{y}, \mathbf{x}A\mathbf{y})$, and $(\mathbf{x}A\mathbf{y}) \succcurlyeq (\mathbf{x}(B \cup C)\mathbf{y})$ imply $(\mathbf{x}A\mathbf{y}) \succcurlyeq (\mathbf{x}B\mathbf{y})$ thanks to NIM.

Proof of the Preadditivity of $\succcurlyeq_I$. Finally, we show that $\succcurlyeq_I$ is preadditive. To this end, consider three subsets of attributes A, B, and C such that $A \cap (B \cup C) = \varnothing$ and $B \succcurlyeq_I C$. By definition of $\succcurlyeq_I$ Equation 2, we know that there exists $\mathbf{x}$ and y such that $x_j \succ_j y_j$ for all $j \in N$ and such that $(\mathbf{x}B\mathbf{y}) \succcurlyeq (\mathbf{x}C\mathbf{y})$. Moreover, we have $C_{\succcurlyeq}(\mathbf{x}(A \cup B)\mathbf{y}, \mathbf{x}(A \cup C)\mathbf{y}) = (A \cup B) \cup (\bar{A} \cap \bar{C}) = ((A \cup B) \cup \bar{A}) \cap ((A \cup B) \cup \bar{C}) = (A \cup B) \cup \bar{C} = B \cup (A \cup \bar{C}) = B \cup \bar{C} = C_{\succcurlyeq}(\mathbf{x}B\mathbf{y}, \mathbf{x}C\mathbf{y})$. Similarly, we have $C_{\succcurlyeq}(\mathbf{x}(A \cup C)\mathbf{y}, \mathbf{x}(A \cup B)\mathbf{y}) = C \cup \bar{B} = C_{\succcurlyeq}(\mathbf{x}C\mathbf{y}, \mathbf{x}B\mathbf{y})$. Hence, we get $C_{\succcurlyeq}(\mathbf{x}(A \cup B)\mathbf{y}, \mathbf{x}(A \cup C)\mathbf{y}) = C_{\succcurlyeq}(\mathbf{x}B\mathbf{y}, \mathbf{x}C\mathbf{y})$, $C_{\succcurlyeq}(\mathbf{x}(A \cup C)\mathbf{y}, \mathbf{x}(A \cup B)\mathbf{y}) = C_{\succcurlyeq}(\mathbf{x}C\mathbf{y}, \mathbf{x}B\mathbf{y})$, and $(\mathbf{x}B\mathbf{y}) \succcurlyeq (\mathbf{x}C\mathbf{y})$, which, by NI, shows that $(\mathbf{x}(A \cup B)\mathbf{y}) \succcurlyeq (\mathbf{x}(A \cup C)\mathbf{y})$ and therefore $(A \cup B) \succcurlyeq_I (A \cup C)$ by Definition 2 of $\succcurlyeq_I$.

Proof of Propositions 4 and 5. By definition, $A \succcurlyeq_I B \Leftrightarrow$ there exists $\mathbf{x}$, $y \in X$ such that for all $j \in N$, $x_j \succ_j y_j$ and $(\mathbf{x}A\mathbf{y}) \succcurlyeq (\mathbf{x}B\mathbf{y})$. Because $(\mathbf{x}A\mathbf{y}) \succcurlyeq (\mathbf{x}B\mathbf{y})$ iff $C_{\succcurlyeq}(\mathbf{x}A\mathbf{y}, \mathbf{x}B\mathbf{y}) \succcurlyeq_M C_{\succcurlyeq}(\mathbf{x}B\mathbf{y}, \mathbf{x}A\mathbf{y})$, we have $A \succcurlyeq_I B \Leftrightarrow C_{\succcurlyeq}(\mathbf{x}A\mathbf{y}, \mathbf{x}B\mathbf{y}) \succcurlyeq_M C_{\succcurlyeq}(\mathbf{x}B\mathbf{y}, \mathbf{x}A\mathbf{y})$. Moreover, $C_{\succcurlyeq}(\mathbf{x}A\mathbf{y}, \mathbf{x}B\mathbf{y}) = A \cup (\bar{A} \cap \bar{B}) = A \cup \bar{B}$ and $C_{\succcurlyeq}(\mathbf{x}B\mathbf{y}, \mathbf{x}A\mathbf{y}) = B \cup (\bar{A} \cap \bar{B}) = B \cup \bar{A}$. Thus, we get $A \succcurlyeq_I B \Leftrightarrow A \cup \bar{B} \succcurlyeq_M B \cup \bar{A}$.

Now, suppose that $A \cup B = N$, i.e., $A \cup \bar{B} = A$ and $B \cup \bar{A} = B$. Then, $A \succ_I B \Leftrightarrow A \succcurlyeq_M B$.

Proof of Proposition 6.

(i) $N >_I \emptyset$ follows from *GDI*.

(ii) The irreflexivity of $>_I$ follows from its definition (strict part of $\succcurlyeq_I$).

(iii) The monotony of $>_I$ is a direct consequence of the monotony of $\succcurlyeq_I$. Indeed, $A >_I B$ implies $A \cup C \succcurlyeq_I B$ by monotony of $\succcurlyeq_I$. If it were the case that $A \cup C \sim_I B$, we would get $B \succcurlyeq_I A$ (by monotony of $\succcurlyeq_I$ again), which contradicts $A >_I B$. So, $A >_I B \Rightarrow A \cup C >_I B$. Similarly, $A >_I B \cup C$ implies $A \succcurlyeq_I B$. If it were the case that $A \sim_I B$, we would get $B \cup C \succcurlyeq_I A$, which contradicts $A >_I B \cup C$. So, $A >_I B \cup C \Rightarrow A >_I B$

(iv) The transitivity of $>_I$ is a direct consequence of the quasi-transivity of $\succcurlyeq$. Now, NIM and DI′ imply that GDI and GDI′ hold. So, we know that there exists $\mathbf{x}, \mathbf{y}, \mathbf{z} \in X$ such that, for all j, for all $\mathbf{w}$ $(\mathbf{x}\{j\}\mathbf{w}) \succ (\mathbf{y}\{j\}\mathbf{w})$, $(\mathbf{y}\{j\}\mathbf{w}) \succ (\mathbf{z}\{j\}\mathbf{w})$ and $(\mathbf{x}\{j\}\mathbf{w}) \succ (\mathbf{z}\{j\}\mathbf{w})$. Suppose that there are three disjoint coalitions A, B, and C such that $A \cup B >_I C$, $A \cup C >_I B$ and build the following alternatives: $\mathbf{a_1} = \mathbf{x}AyBzCw$, $\mathbf{a_2} = \mathbf{y}A\mathbf{z}B\mathbf{x}C\mathbf{w}$, and $\mathbf{a}_3 = \mathbf{z}A\mathbf{x}B\mathbf{y}C\mathbf{w}$. If we denote $D = N\backslash(A \cup B \cup C)$, we get

$$C_{\succcurlyeq}(\mathbf{a}_1, \mathbf{a}_2) = A \cup B \cup D \quad C_{\succcurlyeq}(\mathbf{a}_2, \mathbf{a}_1) = C \cup D$$

$$C_{\succcurlyeq}(\mathbf{a}_2,\mathbf{a}_3) = A \cup C \cup D \quad C_{\succcurlyeq}(\mathbf{a}_3,\mathbf{a}_2) = B \cup D$$

$$C_{\succcurlyeq}(\mathbf{a}_1,\mathbf{a}_3) = A \cup D \qquad C_{\succcurlyeq}(\mathbf{a}_3,\mathbf{a}_1) = B \cup C \cup D$$

Let us apply Theorem 1. Because $\succcurlyeq_I$ is additive and complete, $A \cup B >_I C$ implies that $\mathbf{a}_1 \succ \mathbf{a}_2$ and $A \cup C >_I B$ implies that $\mathbf{a}_2 \succ \mathbf{a}_3$. Thus, by transitivity, we get $\mathbf{a}_1 \succ \mathbf{a}_3$, which is equivalent to $A \cup D >_I B \cup C \cup D$. Because $\succcurlyeq_I$ is additive and complete, this means that $A >_I B \cup C$

(v) In the proof of Proposition 1, it has been shown that there exists $\mathbf{x}, \mathbf{y}$ such that $x_j >_j y_j$, $j = 1, \ldots, n$, and for all A, B, $A \succcurlyeq_I B \Leftrightarrow \mathbf{x}A\mathbf{y} \succcurlyeq \mathbf{x}B\mathbf{y}$. Suppose that there is a A such that $A \sim_I \emptyset$, i.e., $\mathbf{y} \sim \mathbf{x}A\mathbf{y}$. Consider any $j \in A$. By NIM, $\mathbf{x}A\mathbf{y} \sim_I \mathbf{y}$ implies $\mathbf{y} \succcurlyeq \mathbf{x}\{j\}\mathbf{y}$, i.e., by NIM again, $\mathbf{y}\{j\}\mathbf{z} \succcurlyeq \mathbf{x}\{j\}\mathbf{x}$, for all $\mathbf{z}$; so, we get $y_i \succcurlyeq_j x_j$, which contradicts $x_j >_j y_j$. Hence, there can be no $A \sim_I \emptyset$.

Proof of Corollary 1. By Theorem 1 we know that for all $\mathbf{x}, \mathbf{y} \in X$, $\mathbf{x} \succcurlyeq \mathbf{y} \Leftrightarrow C_{\succcurlyeq}(\mathbf{x}, \mathbf{y}) \succcurlyeq_I C_{\succcurlyeq}(\mathbf{y}, \mathbf{x})$. Because $\succcurlyeq$ (and thus $\succcurlyeq_I$) is complete, this can be rewritten as $\mathbf{x} \succ \mathbf{y} \Leftrightarrow C_{\succcurlyeq}(\mathbf{x}, \mathbf{y}) >_I C_{\succcurlyeq}(\mathbf{y}, \mathbf{x})$. Because $\succcurlyeq_I$ is preadditive, $\mathbf{x} \succ \mathbf{y} \Leftrightarrow C_{\succ}(\mathbf{x}, \mathbf{y}) >_I C_{\succ}(y, \mathbf{x})$. Thus, we have to show that the restriction of $>_I$ to disjoint subsets of N is representable by a family of possibility distributions.

The set of properties of $>_I$ obtained in Proposition 6 precisely define what is called an acceptance order and we know by Ref. 38 that such a relation satisfies the postulates of Kraus, Lehmann, and Magidor.[40] So, by Ref. 36 we know that there is a family $\mathcal{F}$ of nondogmatic possibility measures such that

$$\forall A, B \subseteq N \text{ such that } A \cap B = \emptyset, \quad A >_I B \Leftrightarrow \forall\ \pi_i \in \mathcal{F}, \Pi_i(A) > \Pi_i(B)$$

This allows a representation of the restriction of $>_I$ to disjoint sets. Because $\geqslant$ is complete, so is $\geqslant_I$. Therefore, the preadditivity property of $\geqslant_I$ can be rewritten as, for all A, B, $C \subseteq N$

$$A \cap (B \cup C) = \emptyset \Rightarrow (B >_I C \Leftrightarrow A \cup B >_I A \cup C)$$

Hence, the entire $>_I$ can be deduced from its restriction to disjoint sets

$$\forall A, B \subseteq N, A >_I B \Leftrightarrow \forall\ \pi_i \in \mathcal{F}, \Pi_i(A \cap \bar{B}) > \Pi_i(\bar{A} \cap B)$$

Because $\mathbf{x} > \mathbf{y} \Leftrightarrow C_>(\mathbf{x}, \mathbf{y}) >_I C_>(\mathbf{y}, \mathbf{x})$, we can soundly deduce that there is a family $\mathcal{F}$ of possibility measures such that

$$\forall\ \mathbf{x}, \mathbf{y} \in X, \mathbf{x} > \mathbf{y} \Leftrightarrow \forall\ \pi_i \in \mathcal{F}, \Pi_i(C_>(\mathbf{x}, \mathbf{y})) > \Pi_i(C_{\neq}(\mathbf{y}, \mathbf{x}))$$

The completeness of the relation implies the completeness of the marginal utility relations; therefore, $C_>(\mathbf{x}, \mathbf{y})$ is the complement of $C_\geqslant(\mathbf{y}, \mathbf{x})$.

By completeness of $\geqslant$, this relation can be deduced from its strict part ($\mathbf{x} \geqslant \mathbf{y} \Leftrightarrow \ldots$) not ($\mathbf{y} > \mathbf{x}$). This writes

$$\mathbf{x} \geqslant \mathbf{y} \Leftrightarrow \exists\ \pi_i \in \mathcal{F}, \Pi_i(C_>(\mathbf{x}, \mathbf{y})) \geqslant \Pi_i(C_>(\mathbf{y}, \mathbf{x}))$$

It only remains to remark that $\Pi_i(C_>(\mathbf{x}, \mathbf{y})) \geqslant \Pi_i(C_>(\mathbf{y}, \mathbf{x})) \Leftrightarrow \mathrm{Nec}_i(C_\geqslant(\mathbf{x}, \mathbf{y})) \geqslant \mathrm{Nec}_i(C_\geqslant(\mathbf{y}, \mathbf{x}))$.

Proof of Proposition 7. Let $\mathcal{F}$ be a family of nondogmatic possibility distributions on N and $\geqslant_1, \ldots, \geqslant_n$ be a set of weak orders defined on $X_1, \ldots, X_n$, respectively. Consider the preference order $\geqslant$ defined by $\mathbf{x} \geqslant \mathbf{y} \Leftrightarrow$ for all $\pi_i \in \mathcal{F}$ $\mathrm{Nec}_i(C_\geqslant(\mathbf{x}, \mathbf{y})) \geqslant \mathrm{Nec}_i(C_\geqslant(\mathbf{y}, \mathbf{x}))$

- LC directly follows from the completeness of $\geqslant$.
- Proof of quasi-transitivity. In the proof of Proposition 2, we have shown that, if the marginal ranking is a weak order, then $\mathrm{Nec}_i(C_\geqslant(\mathbf{x}, \mathbf{y})) > \mathrm{Nec}_i(C_\geqslant(\mathbf{y}, \mathbf{x}))$ and $\mathrm{Nec}_i(C_\geqslant(\mathbf{y}, \mathbf{z})) > Nec_i(C_\geqslant(\mathbf{z}, \mathbf{y}))$ imply $\mathrm{Nec}_i(C_\geqslant(\mathbf{x}, \mathbf{z})) > Nec_i(C_\geqslant(\mathbf{z}, \mathbf{x}))$. So, for all $\pi_i \in \mathcal{F}$, $\mathrm{Nec}_i(C_\geqslant(\mathbf{x}, \mathbf{y})) \geqslant \mathrm{Nec}_i(C_\geqslant(y, \mathbf{x}))$ and for all $\pi_i \in \mathcal{F}$, $\mathrm{Nec}_i(C_\geqslant(\mathbf{y}, \mathbf{z})) \geqslant \mathrm{Nec}_i(C_\geqslant(\mathbf{y}, \mathbf{z}))$ imply for all $\pi_i \in \mathcal{F}$, $\mathrm{Nec}_i(C_\geqslant(\mathbf{x}, \mathbf{z})) \geqslant \mathrm{Nec}_i(C_\geqslant(\mathbf{z}, \mathbf{x}))$; $\geqslant$ is quasi-transitive.
- Proof of NIM. The relation $\geqslant$ is complete. So, NIM writes for all $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w} \in X$, $[C_\geqslant(\mathbf{x}, \mathbf{y}) \subseteq C_\geqslant(\mathbf{z}, \mathbf{w})$, and $C_\geqslant(\mathbf{y}, \mathbf{x}) \supseteq C_\geqslant(w, z)] \Rightarrow (\mathbf{w} > \mathbf{z} \Rightarrow \mathbf{y} > \mathbf{x})$.
- Consider $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, and $\mathbf{w} \in X$ such that $[C_\geqslant(\mathbf{x}, \mathbf{y}) \subseteq C_\geqslant(\mathbf{z}, \mathbf{w})$ and $C_\geqslant(\mathbf{y}, \mathbf{x}) \supseteq C_\geqslant(\mathbf{w}, \mathbf{z})]$ and $\mathbf{w} > \mathbf{z}$. For all $\pi_i \in \mathcal{F}$ $\mathrm{Nec}_i(C_\geqslant(\mathbf{w}, \mathbf{z})) > \mathrm{Nec}_i(C_\geqslant(\mathbf{z}, \mathbf{w}))$. $C_\geqslant(\mathbf{y}, \mathbf{x}) \supseteq C_\geqslant(\mathbf{w}, \mathbf{z})$ implies that $\mathrm{Nec}_i(C_\geqslant(\mathbf{w}, \mathbf{z})) \leqslant \mathrm{Nec}_i(C_\geqslant(\mathbf{y}, \mathbf{x}))$. $C_\geqslant(\mathbf{x}, \mathbf{y}) \subseteq C_\geqslant(\mathbf{z}, \mathbf{w})$ implies that $\mathrm{Nec}_i(C_\geqslant(\mathbf{z}, \mathbf{w})) \geqslant \mathrm{Nec}_i(C_\geqslant(\mathbf{x}, \mathbf{y}))$. Therefore, we get for all $\pi_i \in \mathcal{F}$ $\mathrm{Nec}_i(C_\geqslant(\mathbf{y}, \mathbf{x})) \geqslant \mathrm{Nec}_i(C_\geqslant(\mathbf{w}, \mathbf{z})) > \mathrm{Nec}_i(C_\geqslant(\mathbf{z}, \mathbf{w})) \geqslant \mathrm{Nec}_i(C_\geqslant(\mathbf{x}, \mathbf{y}))$. This yields for all $\pi_i \in \mathcal{F}$ $\mathrm{Nec}_i(C_\geqslant(\mathbf{y}, \mathbf{x})) > \mathrm{Nec}_i(C_\geqslant(\mathbf{x}, \mathbf{y}))$, i.e., $\mathbf{y} \neq \mathbf{x}$; NIM is satisfied (actually, it is satisfied by any rule based on a family of capacities; it is enough to replace Nec_i by a capacity μ_i in the proof).
- Let us prove that the marginal utility relations built by Equation 3 coincide with the original ones, i.e., that ($x_j \geqslant_j y_j \Leftrightarrow$ for all $z \in X$, $(\mathbf{x}\{j\}\mathbf{z}) \geqslant (\mathbf{y}\{j\}\mathbf{z})$). If it is the case, then DI (resp. DI′) holds iff it holds on the original marginal relations.

Suppose that $x_j \succ_j y_j$. Therefore, whatever z, $C_{\succcurlyeq}(\mathbf{x}\{j\}\mathbf{z}, \mathbf{y}\{j\}\mathbf{z}) = N$ and $C_{\succcurlyeq}(\mathbf{y}\{j\}\mathbf{z}, \mathbf{x}\{j\}\mathbf{z}) = N\backslash\{i\}$. Because the possibility distributions are supposed to be such that for all i, $\pi(i) > 0$ (nondogmatism), it holds that for any π_i, $\mathrm{Nec}_i(N) = 1 > \mathrm{Nec}_i(N\backslash\{i\})$. So, for all $z \in X$, $(\mathbf{x}\{j\}\mathbf{z}) \succ (y\{j\}\mathbf{z}))$. Now, suppose that $x_j \sim_j y_j$. $C_{\succcurlyeq}(\mathbf{x}\{j\}\mathbf{z}, \mathbf{y}\{j\}\mathbf{z}) = N$ and $C_{\succcurlyeq}(\mathbf{y}\{j\}\mathbf{z}, \mathbf{x}\{j\}\mathbf{z}) = N$. Hence, for any π_i, $\mathrm{Nec}_i(C_{\succcurlyeq}(\mathbf{x}\{j\}\mathbf{z}, \mathbf{y}\{j\}\mathbf{z})) = \mathrm{Nec}_i(C_{\succcurlyeq}(\mathbf{y}\{j\}\mathbf{z}, \mathbf{x}\{j\}\mathbf{z}))$, i.e., for all $\mathbf{z} \in X$, $(\mathbf{x}\{j\}\mathbf{z}) \succcurlyeq (\mathbf{y}\{j\}\mathbf{z}))$. Because the marginal ranking are supposed to be complete, the two cases prove that $(x_j \succcurlyeq_j y_j \Leftrightarrow$ for all $\mathbf{z} \in X$, $(\mathbf{x}\{j\}\mathbf{z}) \succcurlyeq (\mathbf{y}\{j\}\mathbf{z}))$.

Proof of Proposition 8. When based on a possibility measure, the decision rule of Definition 2 yields

$$\forall\, \mathbf{x}, \mathbf{y} \in X, \mathbf{x} \succcurlyeq \mathbf{y} \Leftrightarrow \Pi(C_{\succcurlyeq}(\mathbf{x}, \mathbf{y})) \geqslant \Pi(C_{\succcurlyeq}(\mathbf{y}, \mathbf{x}))$$

Suppose that there exists $i \in N$ such that $\pi(i) = 1$ and $x_i \succcurlyeq_i y_i$. Then, obviously,

$$\Pi(C_{\succcurlyeq}(\mathbf{x}, \mathbf{y})) = \max_{i \in C_{\succcurlyeq}(\mathbf{x},\mathbf{y})} \Pi(\{i\}) = 1$$

Thus, whatever $\Pi(C_{\succcurlyeq}(\mathbf{y}, \mathbf{x}))$, $\Pi(C_{\succcurlyeq}(\mathbf{x}, \mathbf{y})) \geqslant \Pi(C_{\succcurlyeq}(\mathbf{y}, \mathbf{x},))$ holds, i.e., $\mathbf{x} \succcurlyeq \mathbf{y}$.

Conversely, suppose not $(x_i \succcurlyeq_i y_i)$ holds for all i such that $\pi(i) = 1$. Because the $\succcurlyeq_i$ are assumed to be complete, this means that for all $i \in N$ such that $\pi(i) = 1$, $y_i \succ_i x_i$. So, $\Pi(C_{\succcurlyeq}(\mathbf{x}, \mathbf{y})) < 1$ and $\Pi(C_{\succcurlyeq}(\mathbf{y}, \mathbf{x})) = 1$; we get $\mathbf{y} \succ \mathbf{x}$. Thus, by contraposition, we have shown that $\mathbf{x} \succcurlyeq \mathbf{y}$ implies that there is a criterion i of possibility 1 such that $x_i \succcurlyeq_i y_i$.

Proof of Proposition 9. Define $\succcurlyeq$ from a possibility measure, as in the previous proof. Note that $\succcurlyeq$ is complete because Π can compare any pair of sets. Moreover, the fact that two different states receive a positive possibility degree implies that the possibility degree of any set of cardinality $n - 1$ is 1, i.e., that for all $j \in N$, $\Pi(N\backslash\{j\}) = 1$.

Consider any $j \in N$ and suppose that there exists $\mathbf{x}$, $\mathbf{y}$ such that $(\mathbf{x}\{j\}\mathbf{y}) \succcurlyeq \mathbf{y}$ and not $(\mathbf{y} \succcurlyeq (\mathbf{x}\{j\}\mathbf{y}))$. The relation $\succcurlyeq$ being complete, this writes: there exists $\mathbf{x}$, $\mathbf{y}$ such that $(\mathbf{x}\{j\}\mathbf{y}) \succ \mathbf{y}$. Four cases are to be considered, depending on how $\succcurlyeq_j$ compares x_j and y_j:

- $x_i \sim_i y_i$. In this case $(\mathbf{x}\{j\}\mathbf{y}) \succ \mathbf{y}$ means $\Pi(N) > \Pi(N)$, which is not possible.
- $x_i \succ_i y_i$. In this case $(\mathbf{x}\{j\}\mathbf{y}) \succ \mathbf{y}$ means $\Pi(N) > \Pi(N\backslash\{j\})$. This leads to a contradiction, because $\Pi(N\backslash\{j\}) = 1$.
- $y_i \succ_i x_i$. In this case $(x\{j\}\mathbf{y}) \succ \mathbf{y}$ means $\Pi(N\backslash\{j\}) > \Pi(N)$, which is not possible.

Therefore, the assumption that there exists $\mathbf{x}$, $\mathbf{y} \in X$, $(\mathbf{x}\{j\}\mathbf{y}) \succ \mathbf{y}$ leads to a contradiction in any case; the satisfaction of DI by a concordance rule based on a possibility measure is incompatible with the existence of two totally possible states.